

QUIZ 16

1. The equation system

$$3x - 7y = 4$$

$$x - 6y = 3$$

$$2x - y = 1$$

has

- a) precise one solution b) infinite many solutions c) two solutions d) no solutions e) None of the others.

Form the augmented matrix + now reduce.

$$\left(\begin{array}{cc|c} 3 & -7 & 4 \\ 1 & -6 & 3 \\ 2 & -1 & 1 \end{array} \right) \xrightarrow{-2R_2+R_3} \left(\begin{array}{cc|c} 3 & -7 & 4 \\ 1 & -6 & 3 \\ 0 & 11 & -5 \end{array} \right) \xrightarrow{-3R_2+R_1}$$

$$\left(\begin{array}{cc|c} 0 & 11 & -5 \\ 1 & -6 & 3 \\ 0 & 11 & -5 \end{array} \right) \xrightarrow{-R_1+R_3} \left(\begin{array}{cc|c} 0 & 11 & -5 \\ 1 & -6 & 3 \\ 0 & 0 & 0 \end{array} \right) \xrightarrow{\frac{6 \cdot R_1}{11 \cdot R_2}}$$

$$\left(\begin{array}{cc|c} 0 & 66 & -30 \\ 11 & -66 & 33 \\ 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1+R_2} \left(\begin{array}{cc|c} 0 & 66 & -30 \\ 11 & 0 & 3 \\ 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{cc|c} 11 & 0 & 3 \\ 0 & 66 & -30 \\ 0 & 0 & 0 \end{array} \right) \rightarrow \begin{aligned} x &= 3/11 \\ y &= -30/66 \end{aligned} \Rightarrow \underline{\text{a) one solution}}$$

2. Let A be a 3×3 matrix with

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Find x_1 from the equation $AX = B$.

- a) 1 b) 2 c) 3 d) 4 e) None of the others.

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow X = A^{-1}B$$

$$\overbrace{X}^{\text{A}^{-1}} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \xrightarrow{\text{ANSWER}} \underline{\underline{x_1 = 3}}$$

3. Which **ONE** of the coefficient matrices below **IS** on reduced form (*land of honey and joy*)

$$(a) \begin{bmatrix} 1 & -3 & 2 & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & -3 & 0 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{PIVOT} \neq 0$$

$$\begin{bmatrix} 1 & -3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) None of them

$$(c) \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)

c) + d) have "bad" staircases.

b) has a nonzero entry above one of the pivots

a) OK

Answer: a)

4. Let a be a number. Find the entry in first row and second column of inverse matrix A^{-1} of the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & a+1 \\ 1 & 1 & a \\ 1 & 0 & a \end{bmatrix}$$

- a) 1 b) $2a$ c) $1-a$ d) a e) None of the others.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & a+1 & 1 & 0 & 0 \\ 1 & 1 & a & 0 & 1 & 0 \\ 1 & 0 & a & 0 & 0 & 1 \end{array} \right) \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & a+1 & 1 & 0 & 0 \\ 1 & 1 & a & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array} \right)$$

\underbrace{A}_{\square}

$$\xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & a & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array} \right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 1 \\ 1 & 1 & a & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array} \right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 1 \\ 1 & 0 & a & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array} \right) \xrightarrow{-aR_1+R_2} \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & -a & 2a & 1-a \\ 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -2 & 1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -a & 2a & 1-a \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right) \quad \text{Answer: } \underline{\underline{2a}}$$

How to "cheat": Notice that if $a=-1$, none of the answers a) b) c) d) are equal. If $a=-1$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{\text{FIND THE INVERSE BY ROW REDUCING OR CALCULATOR}}$$

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \quad \overbrace{\begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}}^{\text{"answer"}}$$

Only b) has the value -2 when $a=-1$.

5. The description of the solution(s) for x_1, x_2, x_3, x_4, x_5 , and x_6 to the equation system

$$x_1 + x_2 + x_4 + 2x_6 = 5$$

$$x_3 + x_4 = 1$$

$$x_5 - x_6 = 1$$

is

- a) No solutions b) $x_1 = 5, x_2 = x_4 = x_6 = 0$, and $x_3 = x_5 = 1$ c) Precisely six solutions d) $x_1 = 5 - x_2 - x_4 - 2x_6, x_3 = 1 - x_4$ and $x_5 = 1 + x_6$, where x_2, x_4 , and x_6 are arbitrary numbers. e) None of the others.

Associated augmented matrix:

$$\left(\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \textcircled{1} & 1 & 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & \textcircled{1} & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 & 1 \end{array} \right) \xrightarrow{\text{TURN THIS BACK INTO A SET OF EQS.}}$$

This is in reduced form. The pivots have been circled. x_1, x_3, x_5 are pivot variables. Solve for these in terms of the free variables.

$$x_1 = 5 - x_2 - x_4 - 2x_6$$

$$x_3 = 1 - x_4$$

$$x_5 = 1 + x_6$$

x_2, x_4, x_6 arbitrary Answer: d)

6. Find the entry in the third row and third column of the matrix $D = AB + C$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & a \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -a & 1 & a & 3 \end{bmatrix},$$

- a) 3 b) $2a$ c) $-a$ d) -1 e) None of the others.

COLUMNS

↓

$$D = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & a \end{bmatrix}}_{\text{Row 3}} \begin{bmatrix} 3 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ -a & 1 & a & 3 \end{bmatrix}}_{(3,3) \text{ entry is } a}.$$

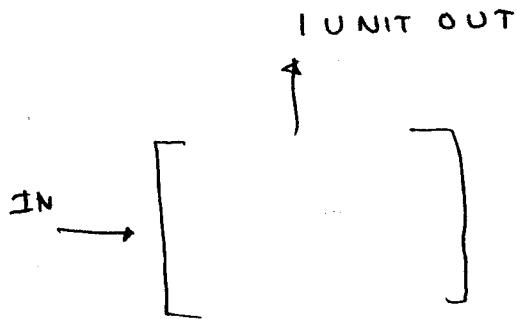
(3,3) entry of this product
 $\rightarrow (0 \ a) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a$

$$D_{3,3} = a + a = \underline{\underline{2a}}$$

7. Suppose that an economy has three goods labeled 1, 2, and 3. The production of 1 unit of good 1 requires .2 units of goods 1 and .4 units of goods 2. The production of good 2 requires .4 units of good 2 and .5 units of good 3. Finally, the production of good 3 requires .1 units of good 2 and .5 units of good 3. Find the technology matrix A for this economy.

- $\begin{matrix} 1 & 2 & 3 \end{matrix}$
- a) $\begin{bmatrix} .2 & .4 & 0 \\ 0 & .4 & .5 \\ 0 & .1 & .5 \end{bmatrix}$ b) $\begin{bmatrix} .2 & 0 & 0 \\ .4 & .4 & .1 \\ 0 & .5 & .5 \end{bmatrix}$ c) $\begin{bmatrix} .2 & .4 & 0 \\ 0 & .4 & .1 \\ 0 & .5 & .5 \end{bmatrix}$ d) $\begin{bmatrix} .2 & 0 & 0 \\ .4 & .4 & .5 \\ 0 & .1 & .5 \end{bmatrix}$

e) None of the others.



8. Suppose that the height measured in meters of trees in a wood in Denmark is normally distributed with mean $\mu = 25.13$ m and standard deviation $\sigma = 7.45$ m. Find the probability that a random selected tree is smaller than 30 m.

- a) .1949 b) .5556 c) .2422 d) .7422 e) None of the others.

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 25.13}{7.45}$$

$$X = 30 \Rightarrow Z = \frac{30 - 25.13}{7.45} \approx .6536$$

$$\begin{aligned} \Pr(X \leq 30) &= \Pr(Z \leq .6536) = \Pr(Z \leq 0) + \Pr(0 \leq Z \leq .6536) \\ &= .5 + .2422 \\ &= .7422 \end{aligned}$$

APPROXIMATE
ANSWER
FROM TABLE

Technically the answer is e), but
d) is very close.

9. Below is a probability density table for a random variable X . Find the expectation $E(X)$ of X .

X	$\Pr[X = x]$
-3	1/4
0	1/3
1	1/6
2	1/4

- (a) 1/12 (b) 1/4 (c) 1 (d) -1/12 (e) None of the others.

$$(-3)\frac{1}{4} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{4} = \underline{\underline{-\frac{1}{12}}}$$

10. Let Z be a standard normal random variable. Find the probability $\Pr[-.27 \leq Z \leq 1.12]$ using the standard normal table.

- a) .3686 b) .1064 c) .4750 d) .5600 e) None of the others.

$$\Pr(-.27 \leq Z \leq 1.12) = \Pr(-.27 \leq Z \leq 0) + \Pr(0 \leq Z \leq 1.12)$$

$$= \Pr(0 \leq Z \leq .27) + \Pr(0 \leq Z \leq 1.12)$$

$$= .1064 \quad + \quad \begin{matrix} \text{LOOK UP ON TABLE} \\ \downarrow \end{matrix} \quad .3686$$

$$= \underline{\underline{.4750}}$$

11. Suppose that an economy has two goods labeled 1 and 2. Let the technology matrix A and the external demand vector D be given by

$$A = \begin{bmatrix} .5 & .2 \\ .5 & .6 \end{bmatrix} \text{ and } D = \begin{bmatrix} 40 \\ 60 \end{bmatrix},$$

respectively. Find the production schedule x_2 for good 2.

- a) 400 units b) 5000 units c) 50 units d) 500 units e) None of the others.

Solve $(I - A)x = D$

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} .5 & .2 \\ .5 & .6 \end{pmatrix} = \begin{pmatrix} .5 & -.2 \\ -.5 & .4 \end{pmatrix}$$

Augment $I - A$ with D & row reduce:

$$\left(\begin{array}{cc|c} .5 & -.2 & 40 \\ -.5 & .4 & 60 \end{array} \right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|c} .5 & -.2 & 40 \\ 0 & .2 & 100 \end{array} \right) \xrightarrow{5R_2} \dots$$

$$\left(\begin{array}{cc|c} .5 & -.2 & 40 \\ 0 & 1 & 500 \end{array} \right) \xrightarrow{x_1 \ x_2} \text{No need to reduce further}$$

$\xrightarrow{x_2 = \underline{\underline{500}}}$

12. Below is a probability density table for a random variable X . The expectation $E(X) = 0$. Find the variance $V(X)$ of X .

X^2	X	$\Pr[X = x]$
16	-4	1/12
1	-1	1/3
0	0	1/6
1	1	1/6
4	2	1/4
		1

- a) $\frac{34}{12}$ b) $\frac{29}{12}$ c) 0 d) -1 e) None of the others.

$$V(X) = E((X - E(X))^2) = E(X^2)$$

\uparrow
 $E(X) = 0$

$$E(X^2) = 16 \cdot \frac{1}{12} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{4} = \frac{17}{6}$$

$$= \frac{34}{12}$$

13. When playing a certain game with your friend you have a probability $p = .54$ of winning in one game. Suppose you play the game 150 times. Let the random variable X be the number of games you won among the 150 games. The expectation of X is $E(X) = 81$. Find the variance $V(X)$ of X .

a) 37.26 b) 36 c) 28 d) 23.84 e) None of the others.

This is a Bernoulli process since the odds are the same for each of the 150 hands. For a Bernoulli process

$$V(X) = n p(1-p) = 150 \cdot .54 \cdot .46 = \underline{\underline{37.26}}$$

↑ ↑
 # of TRIALS Prob(SUCCESS) ON EACH TRIAL

14. Let X be a normal random variable with mean μ and standard deviation σ . What number is closest to the probability $\Pr[X \geq \mu + 2\sigma]$?
- a) .025 b) .95 c) .68 d) .34 e) None of the others.

$$Z = \frac{X - \mu}{\sigma} \quad \text{When } X = \mu + 2\sigma, Z = \frac{\mu + 2\sigma - \mu}{\sigma} = 2.$$

So,

$$\begin{aligned} \Pr(X \geq \mu + 2\sigma) &= \Pr(Z \geq 2) \\ &= \Pr(Z \geq 0) - \Pr(0 \leq Z \leq 2) \\ &= .5 - .4772 = .0228 \end{aligned}$$

15. An urn contains 2 blue Jelly beans, 1 yellow jelly bean, and 3 red jelly beans. Two jelly beans are drawn at random and simultaneously and the colors are noted. Define a random variable X to be the number of blue Jelly beans minus the number of yellow jelly beans. Find the probability density table for X .

X	$\Pr[X = x]$	X	$\Pr[X = x]$	X	$\Pr[X = x]$	X	$\Pr[X = x]$
-1	1/5	-1	1/5	0	1/5	-2	1/3
(a) 0	2/15	(b) 0	1/3	(c) 1	1/3	(d) -1	1/5
1	2/5	1	2/5	2	2/5	1	1/15
2	4/15	2	1/15	3	1/15	2	2/5

(e) None of the others.

Here are the possible outcomes

$$\text{RY} \iff (X = -1) \quad \Pr(X = -1) = \Pr(\text{RY}) = \frac{\binom{3}{1}\binom{1}{1}}{\binom{6}{2}} = \frac{3}{15} = \frac{1}{5}$$

(1 RED + 1 YELLOW)

$$\text{BY or RR} \iff (X = 0) \quad \Pr(X = 0) = \Pr(\text{BY or RR}) = \frac{\binom{2}{1}\binom{1}{1} + \binom{3}{2}}{\binom{6}{2}} = \frac{1}{3}$$

$$\text{BR} \iff (X = 1) \quad \Pr(X = 1) = \Pr(\text{BR}) \\ = \frac{\binom{2}{1}\binom{3}{1}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}$$

$$\text{BB} \iff (X = 2)$$

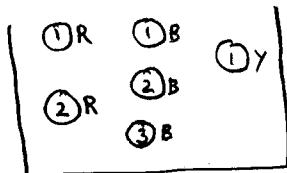
$$\Pr(X = 2) = \Pr(\text{BB}) = \frac{\binom{2}{2}}{\binom{6}{2}} = \frac{1}{15}$$

The — quantities correspond to b.

16. An urn contains two red balls numbered 1 and 2, three blue balls numbered 1, 2, and 3, and one yellow ball with the number 1. Twelve balls are drawn one after the other at random and with replacement. The color and the number of each of the ten balls are noted. Define a random variable X to be the number of odd numbered balls among the twelve drawn balls. Find the expectation $E(X)$ of X .

- a) 4 b) 8 c) 6 d) 12 e) None of the others.

SHOULD BE A 12 NOT 10



This is a Bernoulli process.
The replacement makes the odds the same on every draw.

Of the 6 balls, 4 are odd. So $\Pr(\text{ODD})$ on just 1 draw is $\frac{1}{6} = \frac{2}{3}$. For a Bernoulli process

$$E(X) = n p = 12 \cdot \frac{2}{3} = \underline{\underline{8}}$$

↑
 # TRIALS PROB OF AN ODD
 ON 1 TRIAL

17. A vase contains 4 red and 3 yellow flowers. Two flowers are withdrawn from the vase, one after another, without replacement. What is the probability that they are both red given that they are both the same color?

- a) $2/3$ b) $7/10$ c) $2/9$ d) $4/9$ e) None of the others.

$$\Pr(2R|SC) = \frac{\Pr(2R \cap SC)}{\Pr(SC)}$$

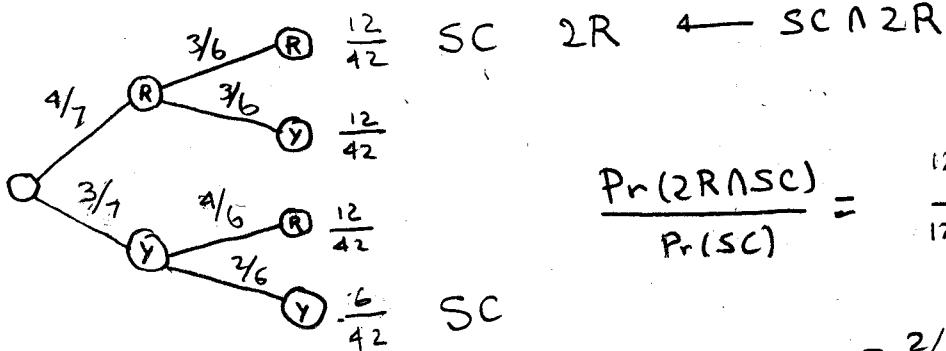
DENOMINATOR: $\Pr(SC) = \frac{\binom{4}{2} + \binom{3}{2}}{\binom{7}{2}}$

$\binom{4}{2}$ ← # ways to draw 2R
 $\binom{3}{2}$ ← # ways to draw 2Y

NUMERATOR $2R \cap SC = 2R$ so $\Pr(2R \cap SC) = \Pr(2R) = \frac{\binom{4}{2}}{\binom{7}{2}}$

$$\Pr(2R|SC) = \frac{\binom{4}{2}/\binom{7}{2}}{\cancel{\binom{4}{2} + \binom{3}{2}}/\binom{7}{2}} = \frac{\binom{4}{2}}{\binom{4}{2} + \binom{3}{2}} = \frac{6}{6+3} = \underline{\underline{2/3}}$$

OR USE A TREE



$$\frac{\Pr(2R \cap SC)}{\Pr(SC)} = \frac{\frac{12}{42}}{\frac{12}{42} + \frac{6}{42}} = \underline{\underline{2/3}}$$

18. Suppose $Pr[A] = .18$, $Pr[B] = .22$, $Pr[C] = .33$, $Pr[A \cap B] = .1$, $Pr[B \cap C] = .11$, $Pr[A \cap C] = .12$, and $Pr[A \cup B \cup C] = .41$. Find $Pr[A \cap B \cap C]$
 a) .04 b) .4 c) .02 d) .17 e) None of the others.

$$Pr(A \cup B \cup C) = .41 =$$

$$= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(B \cap C) - Pr(A \cap C) + Pr(A \cap B \cap C)$$

$$= .18 + .22 + .33 - .1 - .11 - .12 + Pr(A \cap B \cap C)$$

$$\text{So } .41 = .73 - .33 + Pr(A \cap B \cap C)$$

$$\Rightarrow .41 = .40 + Pr(A \cap B \cap C)$$

$$\Rightarrow \underline{\underline{Pr(A \cap B \cap C) = .01.}}$$