SYSTEMATIZING PROBABILITY:

$S = \{s_1, s_2, \ldots, s_n\}$ ⇐ The s_i are outcomes.

$w(s_1)$ $w(s_2)$ $w(s_n)$ ⇐ The $w(s_i)$ are numbers ≥ 0.

ERASE

1

$S = \{s_1, s_2, \ldots, s_n\}$ ⇐ The s_i are outcomes.

$w(s_1)$ $w(s_2)$ $w(s_n)$ ⇐ The $w(s_i)$ are numbers ≥ 0.

If $E = \{s_1, s_3, s_7\} \subset S$, then $Pr[E] = w(s_1) + w(s_3) + w(s_7)$. $Pr[E]$ is the probability that E will occur. Note that

1) $0 \leq Pr[E] \leq 1$

since $w(s_1) + w(s_2) + \cdots + w(s_n) = 1$. Also

2) $Pr[S] = 1$ AND $Pr[\emptyset] = 0$.

If E and F are disjoint ($E \cap F = \emptyset$), then

3) $Pr[E \cup F] = Pr[E] + Pr[F]$

This last statement follows from common sense . . .

ERASE

2

$E = \{s_1, s_5, s_7, s_{22}\}$ $F = \{s_4, s_{59}, s_{78}, s_{212}, s_{21244}\}$

$w(s_1)+w(s_5)+w(s_7)+w(s_{22})$ $w(s_4)+w(s_{59})+w(s_{78})+w(s_{212})+w(s_{21244})$

$Pr[E]$ $Pr[F]$

ERASE

3

$\{s_1, s_5, s_7, s_{22}, s_4, s_{59}, s_{78}, s_{212}, s_{21244}\}$

$w(s_1)+w(s_5)+w(s_7)+w(s_{22})+w(s_4)+w(s_{59})+w(s_{78})+w(s_{212})+w(s_{21244})$

$Pr[E] + Pr[F]$

ERASE

4

$\{s_1, s_5, s_7, s_{22}, s_4, s_{59}, s_{78}, s_{212}, s_{21244}\}$

$w(s_1)+w(s_5)+w(s_7)+w(s_{22})+w(s_4)+w(s_{59})+w(s_{78})+w(s_{212})+w(s_{21244})$

$Pr[E] + Pr[F]$

Clearly, this is also $Pr[E \cup F]$

ERASE

5

$\{s_1, s_5, s_7, s_{22}, s_4, s_{59}, s_{78}, s_{212}, s_{21244}\}$

$w(s_1)+w(s_5)+w(s_7)+w(s_{22})+w(s_4)+w(s_{59})+w(s_{78})+w(s_{212})+w(s_{21244})$

$Pr[E] + Pr[F]$

Clearly, this is also $Pr[E \cup F]$

So, $Pr[E \cup F] = Pr[E] + Pr[F]$ $(E \cap F = \emptyset)$

ERASE

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COMMENT 1: The book adopts 1), 2), and 3) as axioms and then goes on to show that everything you need to know can be derived from these axioms. We will also derive what we need from 1), 2), and 3).

COMMENT 2: One immediate consequence of 3) follows by considering $E \subset S$:

$E \cup E^c = S$ $E \cap E^c = \emptyset$.

So by 3),

$1 = Pr[S] = Pr[E \cup E^c] = Pr[E] + Pr[E^c]$

or

$1 = Pr[E] + Pr[E^c]$ for all $E \subset S$

ERASE

7

RECALL: If A and B are sets, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

The only fact that we used in deriving this was:

$n(A \cup B) = n(A) + n(B)$ for disjoint sets.

We now have the corresponding statement holding true for probabilities

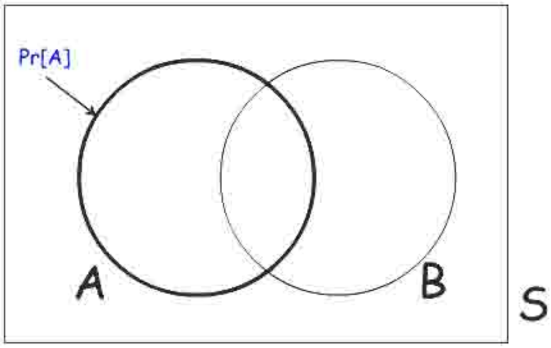
$Pr[A \cup B] = Pr[A] + Pr[B]$ for disjoint sets, so it should be true that

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$ for any sets.

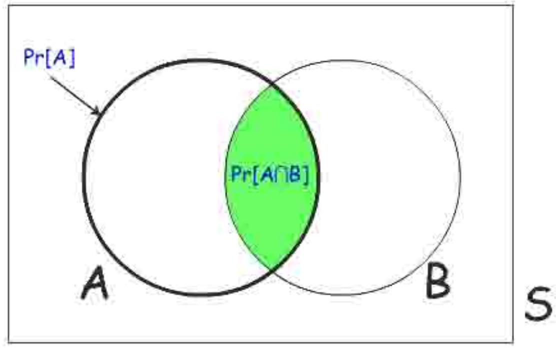
The reasoning is the same: Just label the segments of the Venn diagram with their probabilities -

ERASE

8

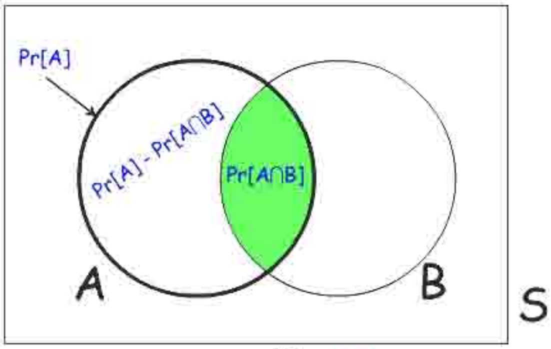


9



So now use $Pr[E \cup F] = Pr[E] + Pr[F]$ for disjoint sets:

10



Noting that the two sets,  and , shown above are disjoint.

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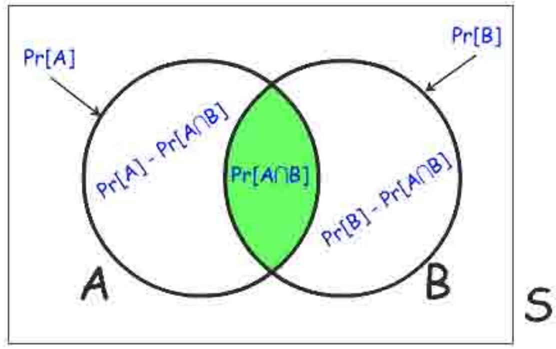
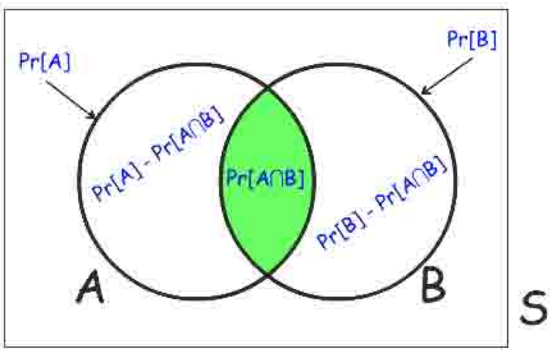


12



$$Pr[A \cup B] = Pr[A] - Pr[A \cap B] + Pr[A \cap B] + Pr[B] - Pr[A \cap B]$$
$$= Pr[A] + Pr[B] - Pr[A \cap B]$$

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All together:

1) $0 \le Pr[E] \le 1$
2) $Pr[S] = 1$ AND $Pr[\varnothing] = 0$.
3) $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$

Minor comment:

Note that $Pr[\varnothing] = 0$ actually follows from $Pr[S] = 1$ and the fact $1 = Pr[E] + Pr[E^c]$ (applied to ø and its complement S).

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EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F]$
b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

15

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F]$
b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

a) $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$

16

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F]$

b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

a) $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$

$.75 = .65 + .4 - Pr[E \cap F]$

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F]$

b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

a) $Pr[E \cup F] = Pr[E] + Pr[F] - Pr[E \cap F]$

$.75 = .65 + .4 - Pr[E \cap F]$

$.75 = 1.05 - Pr[E \cap F]$

$Pr[E \cap F] = .3$

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F] = .3$

b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

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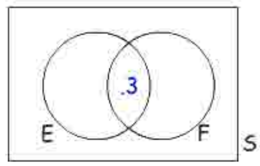
b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

Venn Diagram:



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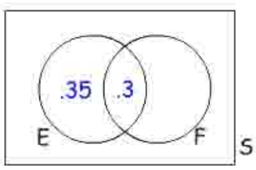

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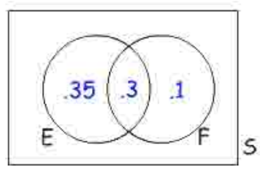

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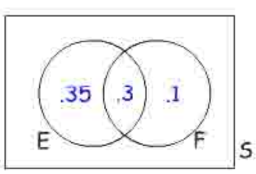
Venn Diagram:



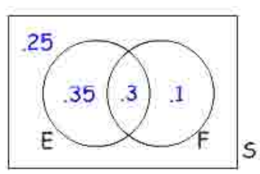
$Pr[S] = 1$

EXAMPLE: Suppose E and F are events in a sample space S with $Pr[E] = .65$, $Pr[F] = .4$ and $Pr[E \cup F] = .75$. Find

a) $Pr[E \cap F] = .3$

b) $Pr[G]$ where G is the set of all outcomes which are in exactly one of the sets E and F.

Venn Diagram:



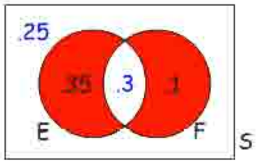
$Pr[S] = 1$

EXAMPLE: Suppose E and F are events in a sample space S with Pr[E] = .65, Pr[F] = .4 and Pr[E ∪ F] = .75. Find

a) Pr[E ∩ F] = .3

b) Pr[G] where G is the set of all outcomes which are in exactly one of the sets E and F.

Venn Diagram: G



Pr[S] = 1

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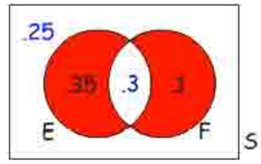
EXAMPLE: Suppose E and F are events in a sample space S with Pr[E] = .65, Pr[F] = .4 and Pr[E ∪ F] = .75. Find

a) Pr[E ∩ F] = .3

b) Pr[G] where G is the set of all outcomes which are in exactly one of the sets E and F.

Venn Diagram: Pr[G] = .35 + .1 = .45



Pr[S] = 1

26

EXAMPLE: A widget manufacturer make widgets. The widgets are often mismanufactured (too large or too small) and/or mislabeled (incorrect label or correct label in wrong place). The probabilities for a randomly selected widget selected from the production line are shown below:

		EVENT	Probability
	F	Acceptable	.72
G	G_1	Too large	.12
G	G_2	Too small	.08
H	H_1	Incorrect label	.15
H	H_2	Correct label/wrong place	.02

27

EXAMPLE: A widget manufacturer make widgets. The widgets are often mismanufactured (too large or too small) and/or mislabeled (incorrect label or correct label in wrong place). The probabilities for a randomly selected widget selected from the production line are shown below:

Added note: Not clearly stated in problem, but we will use it, H_1 and H_2 are disjoint.

		EVENT	Probability
	F	Acceptable	.72
G	G_1	Too large	.12
G	G_2	Too small	.08
H	H_1	Incorrect label	.15
H	H_2	Correct label/wrong place	.02

28

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured

b) Probability widget is not mismanufactured

c) Probability widget is mislabeled

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

29

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured Pr[G]

b) Probability widget is not mismanufactured $Pr[G^c]$

c) Probability widget is mislabeled Pr[H]

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

30

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured Pr[G]

b) Probability widget is not mismanufactured $Pr[G^c]$

c) Probability widget is mislabeled Pr[H]

$Pr[\text{mismanufactured}] = Pr[G] = Pr[G_1 \cup G_2]$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	disjoint
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

31

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured Pr[G]

b) Probability widget is not mismanufactured $Pr[G^c]$

c) Probability widget is mislabeled Pr[H]

$Pr[\text{mismanufactured}] = Pr[G] = Pr[G_1 \cup G_2] = Pr[G_1] + Pr[G_2]$

$G_1 \cap G_2 = \emptyset$

$= .12 + .08 = .20$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	disjoint
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

32

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c]$
c) Probability widget is mislabeled $Pr[H]$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

33

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c]$
c) Probability widget is mislabeled $Pr[H]$

$Pr[\text{NOT mismanufactured}] = Pr[G^c] = 1 - Pr[G] = 1 - .20 = .80$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

34

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H]$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

35

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H]$

$Pr[\text{mislabeled}] = Pr[H] = Pr[H_1 \cup H_2] = Pr[H_1] + Pr[H_2]$

$H_1 \cap H_2 = \emptyset$

$= .15 + .02 = .17$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	disjoint

36

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$

	EVENT	Probability	
F	Acceptable	.72	
G_1	Too large	.12	$G = G_1 \cup G_2$
G_2	Too small	.08	
H_1	Incorrect label	.15	$H = H_1 \cup H_2$
H_2	Correct label/wrong place	.02	

37

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$

	EVENT	Probability
F	Acceptable	.72

38

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured
e) Probability widget is both mislabeled and mismanufactured

	EVENT	Probability
F	Acceptable	.72

39

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured $Pr[G \cup H] = Pr[F^c]$
e) Probability widget is both mislabeled and mismanufactured $Pr[G \cap H]$

	EVENT	Probability
F	Acceptable	.72

40

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured $Pr[G \cup H] = Pr[F^c]$
e) Probability widget is both mislabeled and mismanufactured $Pr[G \cap H]$

$Pr[F^c] = 1 - Pr[F] = 1 - .72 = .28$

	EVENT	Probability
F	Acceptable	.72

41

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured $Pr[G \cup H] = Pr[F^c] = .28$
e) Probability widget is both mislabeled and mismanufactured $Pr[G \cap H]$

	EVENT	Probability
F	Acceptable	.72

42

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured $Pr[G \cup H] = Pr[F^c] = .28$
e) Probability widget is both mislabeled and mismanufactured $Pr[G \cap H]$

$Pr[G \cup H] = Pr[G] + Pr[H] - Pr[G \cap H] \Rightarrow$

	EVENT	Probability
F	Acceptable	.72

43

EXAMPLE: For randomly selected widget, find:

a) Probability widget is mismanufactured $Pr[G] = .20$
b) Probability widget is not mismanufactured $Pr[G^c] = .80$
c) Probability widget is mislabeled $Pr[H] = .17$
d) Probability widget is either mislabeled or mismanufactured $Pr[G \cup H] = Pr[F^c] = .28$
e) Probability widget is both mislabeled and mismanufactured $Pr[G \cap H]$

$Pr[G \cup H] = Pr[G] + Pr[H] - Pr[G \cap H] \Rightarrow$

$.28 = .20 + .17 - Pr[G \cap H] \Rightarrow Pr[G \cap H] = .09$

	EVENT	Probability
F	Acceptable	.72

44

45

46

47

48