COMBINATIONS:

DEFINITION: A combination is selection of r elements from a set of n elements made without regard to order and without repetition.

Note 1: "without repetition" means that once an element is selected then it cannot be selected again.

Note 2: "without regard to order" A prime example of this is a poker hand. The cards are dealt, but the order that the cards were received has no bearing on the strength of the hand.

1

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION; Consider the set

$\{S_1,S_2,S_3,S_4,S_5,S_6,S_7,S_8,S_9,S_{10},S_{11},S_{12}\}$

One particular selection of 3 detectors would be

$\{S_1,S_7,S_{11}\}$

As in the last example about "words", it is easiest to first solve the problem by considering ordered selections. So, first we'll change the problem . . .

2

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and with regard to order. In how many ways can this be done?

3

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and with regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12\cdot11\cdot10 = 1320$

ways to do this. A list of all possibilities would look like:

(S_1,S_8,S_3)
(S_4,S_2,S_1)
(S_1,S_7,S_{11})
(S_8,S_6,S_2)
(S_{11},S_7,S_1)
(S_{12},S_7,S_4)
•
•
etc.

BUT IT'S A LONG LIST

4

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and with regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12\cdot11\cdot10 = 1320$

ways to do this. A list of all possibilities would look like:

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(S_4,S_2,S_1)
(S_1,S_7,S_{11})
(S_8,S_6,S_2)
(S_{11},S_7,S_1)
(S_{12},S_7,S_4)
•
•
etc.

NOW CHANGE THE ( ) BRACKETS TO { } BRACKETS. THIS WILL GIVE US A LIST OF SUBSETS WITH 3 ELEMENTS AS OPPOSED TO ORDERED TRIPLES WITH 3 ELEMENTS.

5

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and with regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12\cdot11\cdot10 = 1320$

ways to do this. A list of all possibilities would look like:

$\{S_1,S_8,S_3\}$
$\{S_4,S_2,S_1\}$
$\{S_1,S_7,S_{11}\}$
$\{S_8,S_6,S_2\}$
$\{S_{11},S_7,S_1\}$
$\{S_{12},S_7,S_4\}$
•
•
etc.

NOW CHANGE THE ( ) BRACKETS TO { } BRACKETS. THIS WILL GIVE US A LIST OF **SUBSETS WITH 3 ELEMENTS** AS OPPOSED TO ORDERED TRIPLES WITH 3 ELEMENTS.

6

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12\cdot11\cdot10 = 1320$

ways to do this. A list of all possibilities would look like:

$\{S_1,S_8,S_3\}$
$\{S_4,S_2,S_1\}$
$\{S_1,S_7,S_{11}\}$
$\{S_8,S_6,S_2\}$
$\{S_{11},S_7,S_1\}$
$\{S_{12},S_7,S_4\}$
•
•
etc.

EVERY POSSIBLE SELECTION OF **3 ELEMENTS** APPEARS ON THIS LIST. HOWEVER, THE ITEMS ARE *"OVER LISTED"* NOW THAT ORDER DOESN'T MATTER.

7

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12\cdot11\cdot10 = 1320$

ways to do this. A list of all possibilities would look like:

$\{S_1,S_8,S_3\}$
$\{S_4,S_2,S_1\}$
$\{S_1,S_7,S_{11}\}$
$\{S_8,S_6,S_2\}$
$\{S_{11},S_7,S_1\}$
$\{S_{12},S_7,S_4\}$
•
•
etc.

QUESTION: How many times does the SET $\{S_1,S_7,S_{11}\}$ appear on the list ?

8

EXAMPLE: From a team of 8 basketball players, 5 are selected to play. In how many teams are possible, IF position assignment doesn't matter.

ERASE

17

EXAMPLE: From a team of 8 basketball players, 5 are selected to play. In how many teams are possible, IF position assignment doesn't matter.

SOLUTION:

$$C(8,5) = \frac{P(8,5)}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 56$$

ERASE

18

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

ERASE

19

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen?

$$C(5,2) = \frac{P(5,2)}{2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

ERASE

20

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen?

$$C(5,2) = \frac{P(5,2)}{2!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

In how many ways can the 2 science courses be chosen?

$$C(4,2) = \frac{P(4,2)}{2!} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

ERASE

21

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen? 10

In how many ways can the 2 science courses be chosen? 6

ERASE

22

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen? 10

In how many ways can the 2 science courses be chosen? 6

Now, for each of the 10 ways to choose the humanities courses, the student has 6 ways to continue in making the choices.

ANSWER: 10 x 6 = 60

ERASE

23

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

CHANGE:

AND ⇒ OR

ERASE

24

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12 \cdot 11 \cdot 10 = 1320$

ways to do this. A list of all possibilities would look like:

$\{S_1,S_8,S_3\}$
$\{S_4,S_2,S_1\}$
$\{S_1,S_7,S_{11}\}$
$\{S_8,S_6,S_2\}$
$\{S_{11},S_7,S_1\}$
$\{S_{12},S_7,S_4\}$
•
•
etc.

QUESTION: How many times does the SET $\{S_1,S_7,S_{11}\}$ appear on the list ?

ANSWER: $3! = 3 \cdot 2 \cdot 1 = 6 = P(3,3)$.
From the set $\{S_1,S_7,S_{11}\}$ choose 3 elements keeping track of order.

9

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12 \cdot 11 \cdot 10 = 1320$

ways to do this. A list of all possibilities would look like:

$\{S_1,S_8,S_3\}$
$\{S_4,S_2,S_1\}$
$\{S_1,S_7,S_{11}\}$
$\{S_8,S_6,S_2\}$
$\{S_{11},S_7,S_1\}$
$\{S_{12},S_7,S_4\}$
•
•
etc.

CONCLUSION: Every set appears on this list 6 times.

10

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: We've already done this. There are

$P(12,3) = 12 \cdot 11 \cdot 10 = 1320$

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$\{S_{12},S_7,S_4\}$
•
•
etc.

CONCLUSION: Every set appears on this list 6 times.

ANSWER TO ORIGINAL PROBLEM:

$$\frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$$

11

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION:

# of ways to select 3 detectors KEEPING TRACK OF ORDER → $12 \cdot 11 \cdot 10$

$$\frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$$

$3 \cdot 2 \cdot 1$ ← Number of ways to order 3 objects

12

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: Let's call this $C(12,3)$. The problem asks . . .

$C(12,3) = ?$

13

EXAMPLE: From a set of 12 smoke detectors, 3 are selected without repetition and without regard to order. In how many ways can this be done?

SOLUTION: Number of ways to select **3 of 12** items keeping track of order.

$$C(12,3) = \frac{P(12,3)}{3!} = \frac{\frac{12!}{(12-3)!}}{3!} = \frac{12!}{3!(12-3)!}$$

Number of ways to order the **3** selected items.

14

GENERAL PROBLEM: In How many ways can r objects be selected from a set of n objects without replacements and without regard to order?

SOLUTION:

$C(n,r) =$

15

GENERAL PROBLEM: In How many ways can r objects be selected from a set of n objects without replacements and without regard to order?

SOLUTION: Number of ways to select **r of n** items keeping track of order.

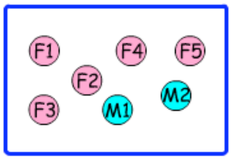
$$C(n,r) = \frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$$

Number of ways to order the **r** selected items.

16

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 2 of the females be selected (without regard to order)?

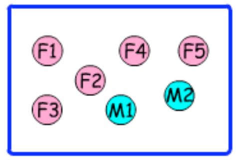
SOLUTION:



From the set of 5 females, 2 will be selected without regard to order.

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 2 of the females be selected (without regard to order)?

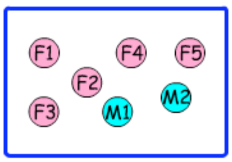
SOLUTION:



From the set of 5 females, 2 will be selected without regard to order.

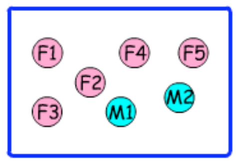
ANSWER: $C(5,2) = \frac{5!}{(5-2)!2!} = 10$

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 2 of the females and 1 of the males be selected (without regard to order)?



EXAMPLE: A room contains 5 females and 2 males. In how many ways can 2 of the females and 1 of the males be selected (without regard to order)?

SOLUTION:

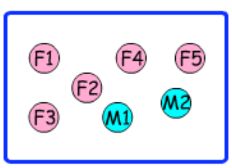


There are 10 ways to select the 2 females. For each one of these 10 ways, there are 2 choices for the male to be selected.

ANSWER: $10 \times 2 = 20$

$= C(5,2) \cdot C(2,1)$

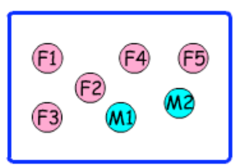
EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)?



EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)?

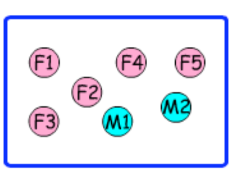
SOLUTION: There are 7 people. There are $C(7,3)$ ways to choose the 3 people.

$$C(7,3) = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$



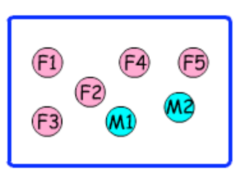
EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)?

35



EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)?

35



EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, OR
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen? 10

In how many ways can the 2 science courses be chosen? 6

Now, the student 16 available choices - if he/she chooses humanities there are 10 choices. However, if science courses are selected, then there are 6 choices.

ANSWER: 10 + 6 = 16

25

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, OR
- 2 general science courses from a list of 4.

In how many ways can this be done?

SOLUTION: In how many ways can the 2 general humanities courses be chosen? 10

In how many ways can the 2 science courses be chosen? 6

Now, the student 16 available choices - if he/she chooses humanities there are 10 choices. However, if science courses are selected, then there are 6 choices.

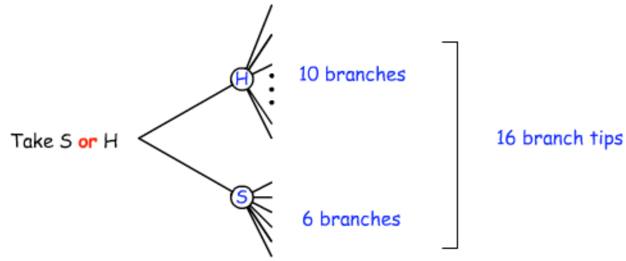
ANSWER: 10 + 6 = 16 HERE'S A TREE . . .

26

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, OR
- 2 general science courses from a list of 4.

In how many ways can this be done?



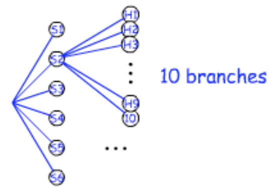
27

EXAMPLE: A student has to take:

- 2 general humanities courses from a list of 5, and
- 2 general science courses from a list of 4.

In how many ways can this be done?

TREE FOR AND:



10 x 6 branch tips total

You see S and H on every trail as opposed to the "or" case where you saw S or H on each path.

28

EXAMPLE: 3 engineers, 2 marketing specialists, and one financial expert, are to be selected from a group of 6 engineers, 5 marketing specialists, and 3 financial experts. In how many ways can the positions be filled?

29

EXAMPLE: 3 engineers, 2 marketing specialists, and one financial expert, are to be selected from a group of 6 engineers, 5 marketing specialists, and 3 financial experts. In how many ways can the positions be filled?

SOLUTION: First, order doesn't matter. All that matters is who gets picked not the order of selection. This is a problem about combinations not permutations.

There are $C(6,3)$ ways to fill the engineering positions.

There are $C(5,2)$ ways to fill the marketing specialist positions.

There are $C(3,1)$ ways to fill the financial expert positions.

30

EXAMPLE: 3 engineers, 2 marketing specialists, and one financial expert, are to be selected from a group of 6 engineers, 5 marketing specialists, and 3 financial experts. In how many ways can the positions be filled?

SOLUTION: First, order doesn't matter. All that matters is who gets picked not the order of selection. This is a problem about combinations not permutations.

There are $C(6,3)$ ways to fill the engineering positions.

There are $C(5,2)$ ways to fill the marketing specialist positions.

There are $C(3,1)$ ways to fill the financial expert positions.

ANSWER: $C(6,3) \cdot C(5,2) \cdot C(3,1) = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \cdot \frac{5 \cdot 4}{2 \cdot 1} \cdot \frac{3}{1} = 600$

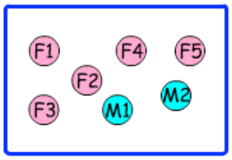
31

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 2 of the females be selected (without regard to order)?

32

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

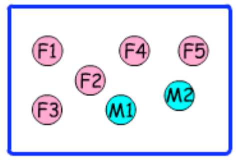
Question 2: How many ways can 3 be selected with at least one male and one female being selected?



41

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

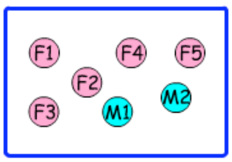
Alternate Question: How many ways can 3 be selected with only males or only females being selected?



42

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Alternate Question: How many ways can 3 be selected with only males or only females being selected?



ONLY FEMALES: C(5,3) = 10

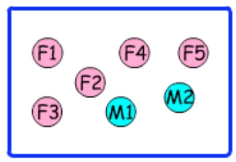
ONLY MALES: C(2,3) = 0

ALL 3 THE SAME: 10

43

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected?

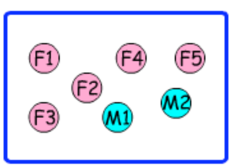


10

44

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected? 10



45

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected? 10

Q2: How many ways with at least one male and one female?

46

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected? 10

Q2: How many ways with at least one male and one female?

  \# ways to pick all male or all female

\+ # ways to pick at least one male and one female

= # ways to pick any 3

47

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected? 10

Q2: How many ways with at least one male and one female?

	# ways to pick all male or all female	10
+	# ways to pick at least one male and one female	?
=	# ways to pick any 3	35

48

EXAMPLE: A room contains 5 females and 2 males. In how many ways can 3 of the people be selected (without regard to order)? 35

Different Question: How many ways can 3 be selected with only males or only females being selected? 10

Q2: How many ways with at least one male and one female?

	# ways to pick all male or all female	10
+	# ways to pick at least one male and one female	?
=	# ways to pick any 3	35

ways to pick at least one male and one female = 25

49

50

Lecture 6