PERMUTATIONS:

DEFINITION: A permutation is an ordered list of elements from a set (without repetition).

Example: Let S = {4,2,3,1}. Then (3,1,2) is a permutation - meaning that 3 is first on the list, 1 is second, and 2 is third. This would be called a permutation of length 3.

Very often, the list will represent a selection process. In this case, it could have happened that 3 was chosen first, then 1, then 2.

1

MAJOR QUESTION: How many permutation are there of a given length (taken from a given set).

EXAMPLE: Let S = {1,2,3,4}. How many permutations are there of length 3?

2

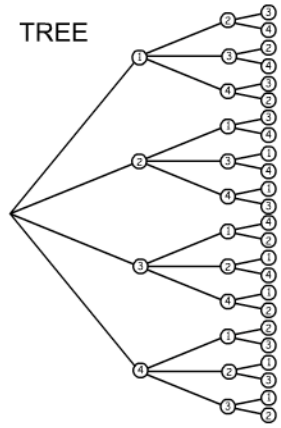
MAJOR QUESTION: How many permutation are there of a given length (taken from a given set).

EXAMPLE: Let S = {1,2,3,4}. How many permutations are there of length 3?

SOLUTION: One way to do this is to draw a tree, and count the number of outcomes. Another way is to use the multiplication principal. At each stage in the tree, the branches split up into the same number of new branches . . .

3

TREE



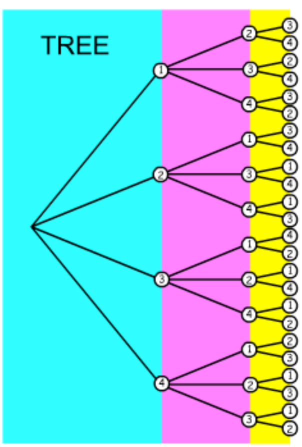
(1,3,4)

Here is one way to select 3 items from the set {1,2,3,4} keeping track of order. First on the list is 1, then 3, then 4.

There are 24 possibilities listed here (count 'em).

Another way to find out is to use the multiplication principal . . .

4



In the blue zone, the single start point breaks up into 4 branches.

In the purple zone, each branch breaks up in to 3 branches.

In the yellow zone, each branch breaks up into 2 branches.

TOTAL BRANCH TIPS:

4 x 3 x 2 = 24

5

EXAMPLE: Let S = {1,2,3,4}. How many permutations are there of length 3?

Perhaps the easiest way to approach this problem is to use the multiplication principal. You have to fill 3 positions using the symbols 1, 2, 3, 4 (with no repetition).

__ __ __

6

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X __ __

There are 4 possibilities for the first position.

7

EXAMPLE: Let S = {1,2,3,4}. How many permutations are there of length 3?

Perhaps the easiest way to approach this problem is to use the multiplication principal. You have to fill 3 positions using the symbols 1, 2, 3, 4 (with no repetition).

4 ways ↘ ↙ 3 ways

X X __

No matter how the first position is filled, there are 3 possibilities for the second position. So there are 4 x 3 ways to fill the first two positions.

8

EXAMPLE: Let $S = \{1,2,3,4\}$. How many permutations are there of length 3?

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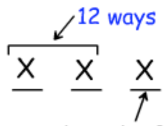
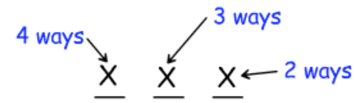


No matter how the first two positions are filled, there are 2 ways to fill in the last position (because there are two numbers left).

EXAMPLE: Let $S = \{1,2,3,4\}$. How many permutations are there of length 3?

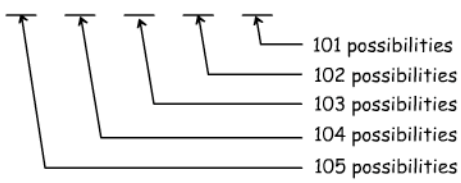
Perhaps the easiest way to approach this problem is to use the multiplication principal. You have to fill 3 positions using the symbols 1, 2, 3, 4 (with no repetition).



Answer: $4 \times 3 \times 2 = 24$.

EXAMPLE: Let S be a set with 105 elements. How many ways can you select 5 items from the set keeping track of order?

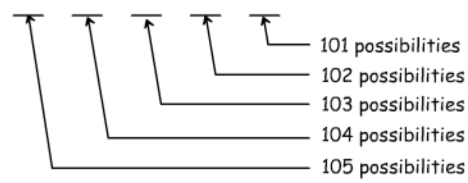
Fill 5 positions:



EXAMPLE: Let S be a set with 105 elements. How many ways can you select 5 items from the set keeping track of order?

Fill 5 positions:

ANSWER: $105 \times 104 \times 103 \times 102 \times 101$



EXAMPLE: Let S be a set with 105 elements. How many ways can you select 5 items from the set keeping track of order?

Fill 5 positions:

ANSWER: $105 \times 104 \times 103 \times 102 \times 101$

__ __ __ __ __

You are invited to draw the tree to verify this. Anybody that does this correctly gets -1 points added to their midterm score.

EXAMPLE: Let S be a set with n elements. How many ways can you select 1 items from the set keeping track of order?

Fill 1 positions:

__

EXAMPLE: Let S be a set with n elements. How many ways can you select 1 items from the set keeping track of order?

Fill 1 positions:

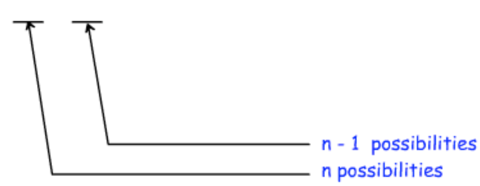
Answer: n

EXAMPLE: Let S be a set with n elements. How many ways can you select 2 items from the set keeping track of order?

Fill 2 positions:

Answer: $n(n-1)$

$=n(n-2+1)$

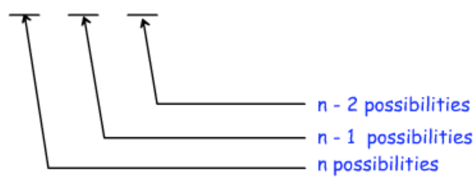


EXAMPLE: Let S be a set with n elements. How many ways can you select 3 items from the set keeping track of order?

Fill 3 positions:

Answer: $n(n-1)(n-2)$

$= n(n-1)(n-3+1)$



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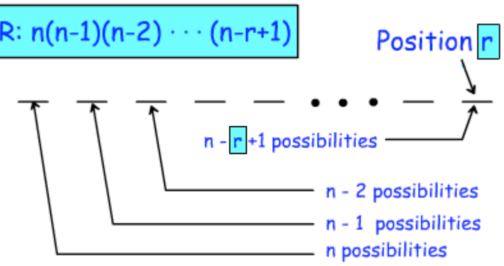
EXAMPLE: Let S be a set with n elements. How many ways can you select r items from the set keeping track of order?

Fill r positions:

ANSWER: $n(n-1)(n-2)\cdots(n-r+1)$



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EXAMPLE: Let S be a set with n elements. How many ways can you select r items from the set keeping track of order?

ANSWER: $n(n-1)(n-2)\cdots(n-r+1)$

There is notation for this (as always).

$P(n,r) \equiv n(n-1)(n-2)\cdots(n-r+1)$ = the number of ways to select r objects from a set of n objects keeping track of order.

19

SOME NOTATION:

$2! = 2\cdot1$

$3! = 3\cdot2\cdot1$

$4! = 4\cdot3\cdot2\cdot1$

$\cdots$

$n! = n(n-1)(n-2)\cdots3\cdot2\cdot1$

20

SOME NOTATION:

$0! = 1$

$1! = 1$

$2! = 2\cdot1$

$3! = 3\cdot2\cdot1$

$4! = 4\cdot3\cdot2\cdot1$

$\cdots$

$n! = n(n-1)(n-2)\cdots3\cdot2\cdot1$

21

$P(n,r) \equiv n(n-1)(n-2)\cdots(n-r+1)$ = the number of ways to select r objects from a set of n objects keeping track of order.

$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) =$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)(n-r-1)\cdots3\cdot2\cdot1}{(n-r)(n-r-1)\cdots3\cdot2\cdot1}$$

$$= \frac{n!}{(n-r)!}$$

22

$P(n,r) \equiv n(n-1)(n-2)\cdots(n-r+1)$ = the number of ways to select r objects from a set of n objects keeping track of order.

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

NOTICE:

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

23

EXAMPLE: Let $S = \{1,2,3,4,5,7,9\}$. Form a four digit number by selecting four of the elements in S without replacement. How many numbers are possible?

SOLUTION: As before, 4 slots are going to be filled.

__ __ __ __

This is just a matter of writing down $P(7,4) = 7\cdot6\cdot5\cdot4 = 840$.

24

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CONTINUATION: How many numbers are possible that contain both even and odd digits?

25

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840 = #(all evens) + #(both evens and odds) + #(all odds)

26

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This is what we want.

27

EXAMPLE: Let $S = \{1,2,3,4,5,7,9\}$.

#(all evens) = 0 There are only two evens in S.

#(all odds) = ?

CONTINUATION: How many numbers are possible that contain both even and odd digits?

840 = #(all evens) + #(both evens and odds) + #(all odds)

This is what we want.

28

EXAMPLE: Let $S = \{1,2,3,4,5,7,9\}$.

#(all evens) = 0 There are only two evens in S.

#(all odds) = ? There are 5 odd numbers in S. We need to pick 4 keeping track of order.

CONTINUATION: How many numbers are possible that contain both even and odd digits?

840 = #(all evens) + #(both evens and odds) + #(all odds)

This is what we want.

29

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#(all odds) = ? There are 5 odd numbers in S. We need to pick 4 keeping track of order.

#(all odds) = $P(5,4) = 5 \cdot 4 \cdot 3 \cdot 2 = 120$

CONTINUATION: How many numbers are possible that contain both even and odd digits?

840 = #(all evens) + #(both evens and odds) + #(all odds)

This is what we want.

30

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CONTINUATION: How many numbers are possible that contain both even and odd digits?

840 = #(all evens) + #(both evens and odds) + #(all odds)

840 = 0 + #(both evens and odds) + 120

31

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CONTINUATION: How many numbers are possible that contain both even and odd digits?

840 = #(all evens) + #(both evens and odds) + #(all odds)

#(both evens and odds) = 840 - 120 = 720

32

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

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SOLUTION: How many ways can the female roles be filled?

Notice that order is imposed by the fact that the roles are distinct (for no other reason that the people must appear at different positions on stage). It may as well be assumed that there is a first female role, a second, third, and fourth female role. So assigning the four roles is the same as selecting 4 of the females KEEPING TRACK OF ORDER. (Assign the first choice to role 1, the second choice to role 2 etc.) This is a problem about permutations.

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SOLUTION: How many ways can the female roles be filled?

From a set of seven people, 4 are selected with regard to order. There are $P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ ways to do this.

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SOLUTION: How many ways can the female roles be filled?

From a set of seven people, 4 are selected with regard to order. There are $P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840$ ways to do this.

How many ways can the male roles be filled?

From a set of eight people, 2 are selected with regard to order. There are $P(8,2) = 8 \cdot 7 = 56$ ways to do this.

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

SOLUTION: How many ways can the female roles be filled?

840

How many ways can the male roles be filled?

56

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

How many ways can the female roles be filled?

840

How many ways can the male roles be filled?

56

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

$$\frac{7}{F}\ \frac{6}{F}\ \frac{5}{F}\ \frac{4}{F}\ \frac{8}{M}\ \frac{7}{M}$$

- 7 ways to fill the first role
- 6 ways to fill the second role for EACH of the 7 previous choices
- 5 ways to fill the third role for EACH of the $7\cdot6$ previous choices
- 4 ways to fill the fourth role for EACH of the $7\cdot6\cdot5$ previous choices
- 8 ways to fill the fifth role for EACH of the $7\cdot6\cdot5\cdot4$ previous choices
- 7 ways to fill the sixth role for EACH of the $7\cdot6\cdot5\cdot4\cdot8$ previous choices

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EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

$$\frac{7}{F}\ \frac{6}{F}\ \frac{5}{F}\ \frac{4}{F}\ \frac{8}{M}\ \frac{7}{M}$$

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- 6 ways to fill the second role for EACH of the 7 previous choices
- 5 ways to fill the third role for EACH of the $7\cdot6$ previous choices
- 4 ways to fill the fourth role for EACH of the $7\cdot6\cdot5$ previous choices
- 8 ways to fill the fifth role for EACH of the $7\cdot6\cdot5\cdot4$ previous choices
- 7 ways to fill the sixth role for EACH of the $7\cdot6\cdot5\cdot4\cdot8$ previous choices

ANSWER: $7\cdot6\cdot5\cdot4\cdot8\cdot7$

50

EXAMPLE: How many 4 letter "words" can be formed using the letters ***GOOD***?

51

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PARTIAL ANSWER: Pretend that the letters all distinct: GO_1O_2D

This is now a pretty standard problem. 4 places

__ __ __ __

are to be filled with four items, thus forming an ordered list of the items G, O_1, O_2, and D.

52

EXAMPLE: How many 4 letter "words" can be formed using the letters ***GOOD***?

PARTIAL ANSWER: Pretend that the letters are all distinct - $G\ O_1\ O_2\ D$:

This is now a pretty standard problem. 4 places

__ __ __ __

are to be filled with four items, thus forming an ordered list of the items G, O_1, O_2, and D. There are $4\cdot3\cdot2\cdot1 = 24$ ways to do this.

53

EXAMPLE: How many 4 letter "words" can be formed using the letters ***GOOD***?

PARTIAL ANSWER: HERE THEY ARE:

GO_1O_2D	O_1GO_2D	O_1O_2GD	O_1O_2DG
GO_2O_1D	O_2GO_1D	O_2O_1GD	O_2O_1DG
GO_1DO_2	O_1GDO_2	O_1DGO_2	O_1DO_2G
GO_2DO_1	O_2GDO_1	O_2DGO_1	O_2DO_1G
GDO_1O_2	DGO_1O_2	DO_1GO_2	DO_1O_2G
GDO_2O_1	DGO_2O_1	DO_2GO_1	DO_2O_1G

Notice: If the 1's and 2's are erased, every "word" appears twice, because every word appears with O_1O_2 and O_2O_1.

54

EXAMPLE: How many 4 letter "words" can be formed using the letters ***GOOD***?

PARTIAL ANSWER: HERE THEY ARE:

GO O D	O GO D	O O GD	O O DG
GO O D	O GO D	O O GD	O O DG
GO DO	O GDO	O DGO	O DO G
GO DO	O GDO	O DGO	O DO G
GDO O	DGO O	DO GO	DO O G
GDO O	DGO O	DO GO	DO O G

Notice: If the 1's and 2's are erased, every "word" appears twice, because every word appears with O_1O_2 and O_2O_1.

ANSWER: 24/2 = 12

55

EXAMPLE: How many 4 letter "words" can be formed using the letters ***GOOD***?

This last bit of reasoning could have been done without writing out all the permutations of G, O_1, O_2, and D.

Notice: If the 1's and 2's are erased, every "word" appears twice, because every word appears with O_1O_2 and O_2O_1.

ANSWER: 24/2 = 12

56

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

How many ways can the female roles be filled? 840

How many ways can the male roles be filled? 56

41

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How many ways can the female roles be filled? 840

How many ways can the male roles be filled? 56

So, to fill all of the roles there are 840 ways to fill the female roles:

_ _ _ _ _ _
F F F F M M

42

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_ _ _ _ _ _
F F F F M M

For each one of these 840 ways, there are 56 ways to proceed and fill the male roles.

43

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So, to fill all of the roles there are 840 ways to fill the female roles: ← 56 ways

_ _ _ _ _ _
F F F F M M

For each one of these 840 ways, there are 56 ways to procede and fill the male roles. **ANSWER: 840 x 56 = 47,040**

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EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

7
F F F F M M

- 7 ways to fill the first role

45

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

7 6
F F F F M M

- 7 ways to fill the first role
- 6 ways to fill the second role for EACH of the 7 previous choices

46

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

7 6 5 4
F F F F M M

- 7 ways to fill the first role
- 6 ways to fill the second role for EACH of the 7 previous choices
- 5 ways to fill the third role for EACH of the $7 \cdot 6$ previous choices
- 4 ways to fill the fourth role for EACH of the $7 \cdot 6 \cdot 5$ previous choices

47

EXAMPLE: 7 females and 8 males try out for a play. 4 female roles need to be filled and 2 male roles need to be filled. How many can this be done? (Assume that there are no cross-dressers.)

QUICK WAY:

7 6 5 4 8
F F F F M M

- 7 ways to fill the first role
- 6 ways to fill the second role for EACH of the 7 previous choices
- 5 ways to fill the third role for EACH of the $7 \cdot 6$ previous choices
- 4 ways to fill the fourth role for EACH of the $7 \cdot 6 \cdot 5$ previous choices
- 8 ways to fill the fifth role for EACH of the $7 \cdot 6 \cdot 5 \cdot 4$ previous choices

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EXAMPLE: How many 4 letter "words" can be formed using the letters **GOOD**?

There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to arrange the letters G, O_1, O_2, and D to form a 4 letter word. Half of these words will look like $\cdots O_1 \cdots O_2 \cdots$ and half will look like $\cdots O_2 \cdots O_1 \cdots$.

ANSWER: $\dfrac{4 \cdot 3 \cdot 2 \cdot 1}{2} = \dfrac{4!}{2!} = 12$

57

EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

58

EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

First, what is the answer if the letters are distinct?

$P_1E_1E_2P_2E_3R$

ANSWER: $P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

59

EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

First, what is the answer if the letters are distinct?

$P_1E_1E_2P_2E_3R$

ANSWER: $P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$. Suppose a list is made and one word on the list is $E_1RP_1E_2P_2E_3$. When all of the 1's, 2's, and 3's are erased, how many times will the word ERPEPE appear on the list?

All words of the form E.RP.E.P.E. will end up looking like ERPEPE when the 1's, 2's, and 3's are erased.

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EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

E.RP.E.P.E. The letters E_1, E_2, E_3 have to be placed in the 3 "E" positions. There are 3! ways to do this (3 items are placed keeping track of order). For each one of these ways, there will be 2! ways to place the 1, and 2 subscripts on the P's.
CONCLUSION: The E's can be arranged 3! ways and the P's can be arranged 2! ways. The word ERPEPE will appear $3! \cdot 2! = 6 \times 2 = 12$ times on the list, as will every word.

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EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

Here are the 12 ways E.RP.E.P.E. appears on the list.

$E_1RP_1E_2P_2E_3$ $E_1RP_1E_3P_2E_2$ $E_2RP_1E_1P_2E_3$
$E_2RP_1E_3P_2E_1$ $E_3RP_1E_1P_2E_2$ $E_3RP_1E_2P_2E_1$

$E_1RP_2E_2P_1E_3$ $E_1RP_2E_3P_1E_2$ $E_2RP_2E_1P_1E_3$
$E_2RP_2E_3P_1E_1$ $E_3RP_2E_1P_1E_2$ $E_3RP_2E_2P_1E_1$

62

EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

ANSWER:

$$\frac{6!}{2!3!}$$

6! ← Number of words on list using 6 different letters.

2!3! ← Number of times each word will appear on the list after erasing subscripts.

63

EXAMPLE: How many 6 letter "words" can be formed using the letters **PEEPER**?

ANSWER:

$$\frac{6!}{2!3!}$$

6! ← Number or ways to arrange 6 different letters.

2! ← Number of ways to arrange the 2 P's.

3! ← Number of ways to arrange the 3 E's.

64

EXAMPLE: How many 6 letter "words" can be formed using the letters PEEPER?

ANSWER:

$$\frac{6!}{2!3!} = \frac{720}{2 \cdot 6} = 60$$

ERASE

65

Lecture 5