PROBABILITIES and EVENTS

DEFINITION: An EVENT is a subset of a sample space.

This is odd, we have invented a new name for something for which we already have a name. Consider the following example . . .

EXAMPLE: Roll a die and record the result - as to the number shown on the top die face. The sample space for this procedure is:

$$S = \{1,2,3,4,5,6\}$$

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Suppose that you win $1 million if a 4, 5, or 6 is rolled. So the event of a 4, 5, 6 being rolled is of monumental importance to you. This corresponds naturally to the set $\{4,5,6\}$. If the outcome is in this set, then you win. Furthermore, the set $\{4,5,6\}$ corresponds to or represents all of the information about the event that you are interested in. By convention $\{4,5,6\}$ called an event. It "is" the event of rolling a 4, 5, or 6.

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. Recall what the sample looks like:

Let

$$R = \{H,T\}$$

and let

$S = R \times R \times R =$

$\{(H,H,H), (H,H,T), (H,T,H), (T,H,H), (H,T,T), (T,H,T), (T,T,H), (T,T,T)\}$

or (slightly different notation):

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. How many elements are in the sample space?

SOLUTION: A typical element in the sample space will look like TTH, representing a tails then a tails then a heads. In general 3 positions have to be filled with T or H.

2 possibilities T or H — 2 possibilities T or H — 2 possibilities T or H

__ __ __

$$2 \times 2 \times 2 = 8$$

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. Draw a tree.

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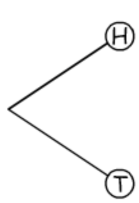

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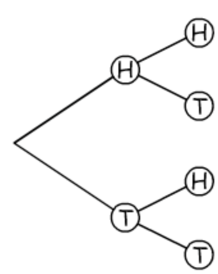

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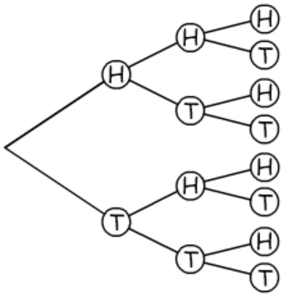
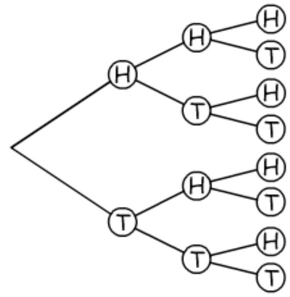


HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. Let A be the event of getting two or more heads. Indicate A on the tree and describe A as a subset of S.



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

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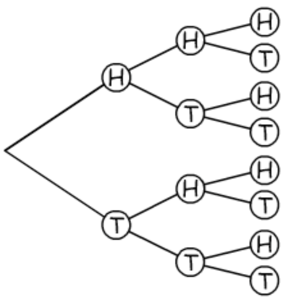

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

A

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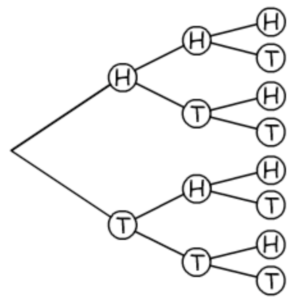
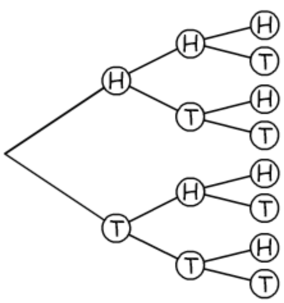


HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

A

A ={HHH, HHT, HTH, THH}

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. Let B be the event of all tosses the same. Indicate B on the tree and describe B as a subset of S.



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

A

B

A ={HHH, HHT, HTH, THH}

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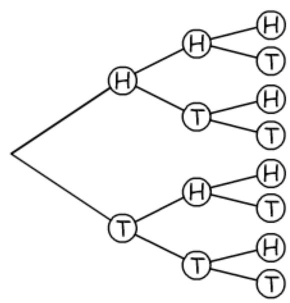


HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

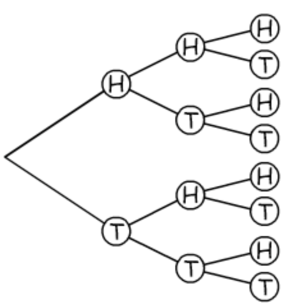
A

B

A ={HHH, HHT, HTH, THH}

B ={HHH, TTT}

EXAMPLE: Flip a coin 3 times. Record the results (heads or tails) in the order that they occur. Indicate $A \cap B$ on the tree and describe $A \cap B$ as a subset of S.



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

A

B

$A \cap B$

A ={HHH, HHT, HTH, THH}

B ={HHH, TTT}

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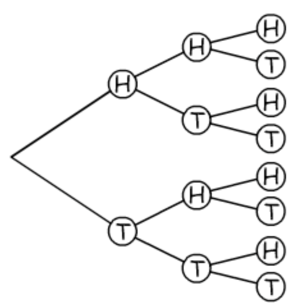


HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$A \cap B$ = {HHH}

A

B

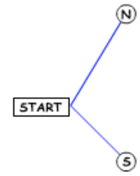
$A \cap B$

A ={HHH, HHT, HTH, THH}

B ={HHH, TTT}

EXAMPLE: Telephone sales: Calls are made to prospective customers. Each call results in a sale S or a no sale N. The calls are made one after another until either there are 2 no sales or a total of 4 calls have been made. What is the sample space?

HINT: FIRST DRAW THE TREE



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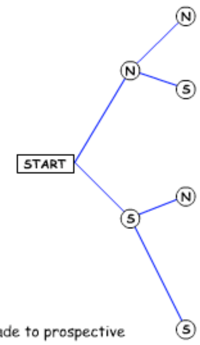

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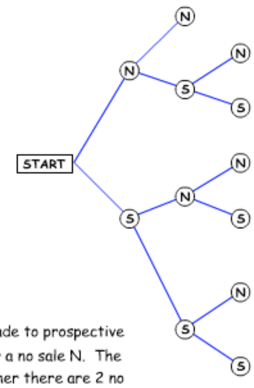

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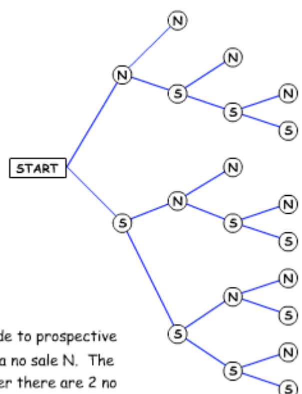

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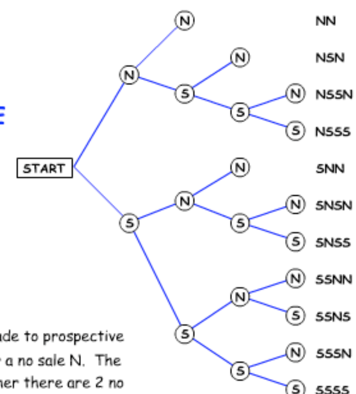
THIS IS THE TREE



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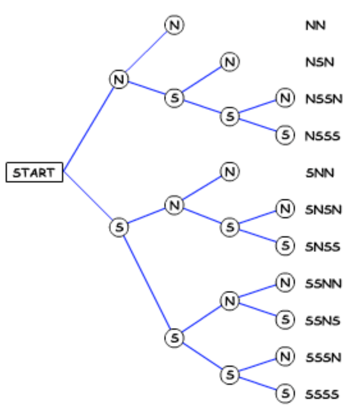


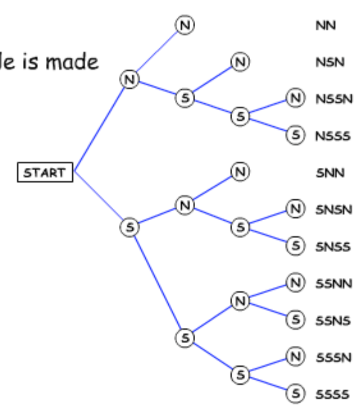
The sample space is:

S = {NN,NSN,NSSN,NSSS,SNN,SNSN,SNSS,SSNN,SSNS,SSSN,SSSS}

Let

E=event that exactly one sale is made

F=exactly one "no sale"



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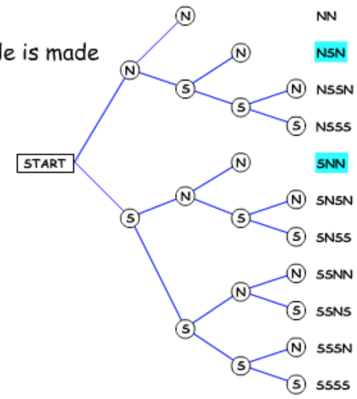

Here's E

S = {NN,NSN,NSSN,NSSS,SNN,SNSN,SNSS,SSNN,SSNS,SSSN,SSSS}

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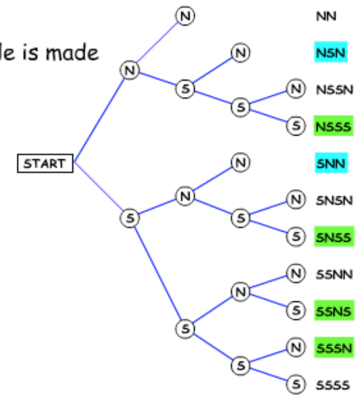

Here's E

Here's F

S = {NN,NSN,NSSN,NSSS,SNN,SNSN,SNSS,SSNN,SSNS,SSSN,SSSS}

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Let
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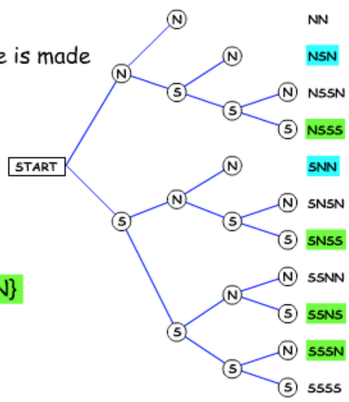

E={NSN,SNN}

F={NSSS,SNSS,SSNS,SSSN}

S = {NN,NSN,NSSN,NSSS,SNN,SNSN,SNSS,SSNN,SSNS,SSSN,SSSS}

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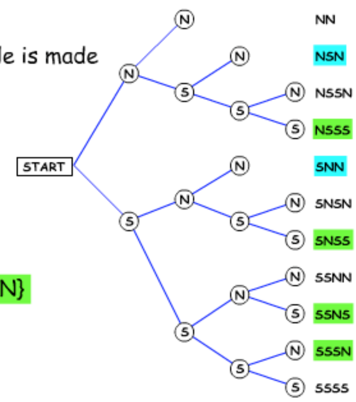
Let
E=event that exactly one sale is made
F=exactly one "no sale"



E={NSN,SNN}

F={NSSS,SNSS,SSNS,SSSN}

NOTE: $E \cap F = \emptyset$.

S = {NN,NSN,NSSN,NSSS,SNN,SNSN,SNSS,SSNN,SSNS,SSSN,SSSS}

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PROBABILITIES:

Given a sample space, $S = \{s_1, s_2, \ldots, s_n\}$, a probability is an assignment of numbers $w(s_1), w(s_2), \ldots, w(s_n)$ to each of the outcomes in $s_1, s_2, \ldots, s_n \in S$, such that $0 \le w(s_i) \le 1$ for $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} w(s_i) = w(s_1) + w(s_2) + \ldots + w(s_n) = 1$$

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Given a sample space, $S = \{s_1, s_2, \ldots, s_n\}$, a probability is an assignment of numbers $w(s_1), w(s_2), \ldots, w(s_n)$ to each of the outcomes in $s_1, s_2, \ldots, s_n \in S$, such that $0 \le w(s_i) \le 1$ for $i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} w(s_i) = w(s_1) + w(s_2) + \ldots + w(s_n) = 1$$

The exact values of the $w(s_i)$ are the whole deal, and $w(s_i)$ is called the probability that the outcome s_i will occur.

Definition: If $E \subset S$ then $Pr[E]$, the probability of E, is the sum of the probabilities of each of the outcomes in E.

$$\sum_{s_i \in E} w(s_i) = Pr[E]$$

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$S = \{s_1, s_2, \ldots, s_n\}$ $\Leftarrow$ The s_i are outcomes.

$w(s_1)$ $w(s_2)$ $w(s_n)$ $\Leftarrow$ The $w(s_i)$ are numbers.

EXAMPLE:
If $E = \{s_1, s_3, s_{11}\} \subset S$ then $Pr[E] = w(s_1) + w(s_3) + w(s_{11})$.

NOTICE:

$$0 \le w(s_1) + w(s_3) + w(s_{11}) \le w(s_1) + w(s_2) + \ldots + w(s_n) \le 1$$

SO:

$$0 \le Pr[E] \le 1$$

FURTHERMORE: $Pr[S] = 1$ AND $Pr[\emptyset] = 0$.

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EXAMPLE: Roll a six sided unfair die (and record the number on the top die face. The sample space for this procedure is:

$$S = \{1,2,3,4,5,6\}$$

Suppose $w(1) = 1/2$, $w(2) = 1/10$, $w(3) = 1/10$, $w(4) = 1/10$, $w(5) = 1/10$, and $w(6) = 1/10$. Let $E = \{1,3,5\}$.

What is $Pr[E]$?

$$Pr[E] = w(1) + w(3) + w(5) = 1/2 + 1/10 + 1/10 = 7/10$$

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EQUALLY LIKELY EVENTS:

Very often an assumption of equally likely events can or will be made. This means that if $S = \{s_1, s_2, \ldots, s_n\}$, then

$$w(s_1) = w(s_2) = \ldots = w(s_n).$$

Since

$$w(s_1) + w(s_2) + \ldots + w(s_n) = 1$$

it follows that

$$w(s_1) = w(s_2) = \ldots = w(s_n) = 1/n.$$

NOTICE: If $E \subset S$ and if E has m points in it, $n(E) = m$, then

$$\Pr[E] = \underbrace{1/n + 1/n + \ldots + 1/n}_{\text{m terms}} = \frac{m}{n} = \frac{n(E)}{n(S)}$$

bad notation

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EXAMPLE: Roll two fair dice - one red, one green. Record the numerical result of each die. What is the probability of rolling a sum of 9?

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EXAMPLE: Roll two fair dice - one red, one green. Record the numerical result of each die. What is the probability of rolling a sum of 9? **Here "is" the sample space.**

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

36 elements total

All of these outcomes are equally likely. (2,6) is just as likely as (5,3). (3,4) is just as likely as (5,5).

35

EXAMPLE: Roll two fair dice - one red, one green. Record the numerical result of each die. What is the probability of rolling a sum of 9?

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	**(3,6)**
4	(4,1)	(4,2)	(4,3)	(4,4)	**(4,5)**	(4,6)
5	(5,1)	(5,2)	(5,3)	**(5,4)**	(5,5)	(5,6)
6	(6,1)	(6,2)	**(6,3)**	(6,4)	(6,5)	(6,6)

36 elements total

Here are the outcomes that correspond to rolling a "9." There are 4 such outcomes. The probability of rolling a 9 is:

$$4/36 = 1/9$$

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EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

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EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

The sample space representing the possible results from the process of drawing one card looks like:

S = {A♥ , A♦ , A♣ , A♠ , 2♥ , 2♦ , 2♣ , 2♠ , 3♥ , 3♦ , 3♣ , 3♠ , . . . K♥ , K♦ , K♣ , K♠ }

where 2♥ represents, for example, the outcome of drawing the two of hearts. Each of the outcomes is just as likely as any other. There are 52 possible outcomes. TWO of these outcomes represent drawing a red 4, namely 4♥ , 4♦ .

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EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

So the probability of drawing a red 4 is: $2/52 = 1/26$.

S = {A♥ , A♦ , A♣ , A♠ , 2♥ , 2♦ , 2♣ , 2♠ , 3♥ , 3♦ , 3♣ , 3♠ , . . . K♥ , K♦ , K♣ , K♠ }

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39

EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

FORMAL WAY TO SAY THE SAME THING:

Let E = the event of drawing a red 4. Then

E = {4♥ , 4♦ }

and $n(E) = 2$. So,

$$\Pr[E] = \frac{n(E)}{n(S)} = \frac{2}{52}$$

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EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

ANOTHER WAY:

Let

S = {R1,B1,R2,B2,R3,B3, . . . ,RK,BK}

where, for example R1 indicates the outcome of drawing a red ace, B3 indicates the outcome of a black 3, and RK represents the outcome of a red king. As far as we are concerned, color and value is the only information that matters. You are free to choose whatever sample space you want. This choice works well since the outcomes are ALL equally likely.

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EXAMPLE: Draw a card from a 52 card deck. Assuming that drawing one card is just as likely as another, what is the probability of drawing a red 4?

ANOTHER WAY:

Let

S = {R1,B1,R2,B2,R3,B3, . . . ,RK,BK}

And let F be the event of drawing a red 4. Then F = {R4}, and n(F) = 1. Also n(S) = 26, so

$$Pr[F] = \frac{n(F)}{n(S)} = \frac{1}{26}$$

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MAIN POINT: If S is a sample space and E is an event in that sample space, then under an assumption of equally likely outcomes:

$$Pr[E] = \frac{n(E)}{n(S)}$$

It now becomes important to be able to calculate n(E) and n(S). In general, we will develop ways to do this. These methods will be called "counting arguments" since they "count" the number of elements in a set.

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