$$P^n = \begin{bmatrix} P_{1,1}(n) & P_{1,2}(n) & P_{1,3}(n) \\ P_{2,1}(n) & P_{2,2}(n) & \boxed{P_{2,3}(n)} \\ P_{3,1}(n) & P_{3,2}(n) & P_{3,3}(n) \end{bmatrix}$$

probability that after n transitions, the system will be in state 3 if it starts in state 2.

$(P^n)_{i,j} = P_{i,j}(n) =$ probability that the system is in state j after n transitions if it starts out in state i.

1

Lecture 31

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 3/8 & 1/8 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 3/8 & 1/4 & 3/8 \end{bmatrix}$$

NO ZEROS ⇒ P REGULAR

2

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$$

regular?

$$P^3 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

NO ZEROS ⇒ P REGULAR

3

Definition: A vector $w = [w_1, w_2, \ldots, w_n]$ is called a probability vector if

1) $w_1 + w_2 + \ldots + w_n = 1$.

2) $w_i \geq 0$ for i = 1, 2, . . . , n.

In the context of Markov chains, these are sometimes called state vectors.

4

Definition: A vector $w = [w_1, w_2, \ldots, w_n]$ is called a probability vector if

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2) $w_i \geq 0$ for i = 1, 2, . . . , n.

In the context of Markov chains, these are sometimes called state vectors.

Example: [.2, .2, .6] In general we'll use these to represent the probabilities of a system to be in different states. In this case Pr[System in state 1] = .2, Pr[State 2] = .2, Pr[State 3] = .6.

5

Definition: A Markov chain is said to have a stable vector $w = [w_1, w_2, \ldots, w_n]$ if w is a state vector and

1) $wP = w$ (P = transition matrix for the chain)

2) Given any state vector $z = [z_1, z_2, \ldots, z_n]$,

$zP^n \rightarrow w$

This last line means that the entries of the vector zP^n get closer and closer (eventually) to the entries of w as n increases.

6

Definition: A Markov chain is said to have a stable vector $w = [w_1, w_2, \ldots, w_n]$ if w is a state vector and

1) $wP = w$ (P = transition matrix for the chain)

2) Given any state vector $z = [z_1, z_2, \ldots, z_n]$,

$zP^n \rightarrow w$

Comment: If there is a stable vector, it is the only state vector with $wP = w$. If $qP = q$ and q is a state vector, then $qP^n \rightarrow w$ since w is stable. But $qP^n = q$, so $q \rightarrow w$, which implies q = p.

7

Theorem: Every regular Markov chain has a stable vector.

There is a state vector w such that $wP = w$, and it is the only vector with this property.

Furthermore, given any state vector z, $zP^n \rightarrow w$.

8

THE TRANSPOSE OF A MATRIX:

$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ Here is a 3 x 2 matrix. It has a transpose called A^T.

$A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ A^T is obtained by interchanging rows and columns - first row of A is the first column of A^T, second row of A is the second column of A^T, etc.

9

THE TRANSPOSE OF A MATRIX:

Fact: $(AB)^T = B^T A^T$

10

Lecture 31

FINDING A STABLE VECTOR:

$wP = w$

$(wP)^T = w^T$

$P^T w^T = w^T$

$P^T w^T - w^T = \mathbf{0}$

$P^T w^T - I w^T = \mathbf{0}$

$(P^T - I) w^T = \mathbf{0}$

11

FINDING A STABLE VECTOR:

$(P^T - I) w^T = \mathbf{0}$

12

FINDING A STABLE VECTOR:

$(P^T - I) w^T = \mathbf{0}$ This has to be solved for w (or w^T).

The technique for this is already developed.

Solve: $Ax = b$ (in this case $b = \mathbf{0}$)

Form the augmented matrix

$[A|b]$

and row reduce.

13

EXAMPLE: Find a stable vector, if it exists, for

$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$(P^T - I) w^T = \mathbf{0}$

14

EXAMPLE: Find a stable vector, if it exists, for

$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$(P^T - I) w^T = \mathbf{0}$

$P^T - I = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix}$

15

EXAMPLE: Find a stable vector, if it exists, for

$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

$(P^T - I) w^T = \mathbf{0}$

$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix}$

16

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad -1/2\, w_1 + 0w_2 + 1/2\, w_3 = 0$$

THIS CORRESPONDS TO A SYSTEM OF 3 EQUATIONS AND 3 UNKNOWNS.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad -1/2\, w_1 + 0w_2 + 1/2\, w_3 = 0$$

THERE IS ONE MORE EQUATION NOT SHOWN HERE

$w_1 + w_2 + w_3 = 1$ SINCE w IS A STATE VECTOR.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$w_1 + w_2 + w_3 = 1$ SINCE w IS A STATE VECTOR.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

FORM THE AUGMENTED MATRIX

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{array}{cccc} w_1 & w_2 & w_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

NOW ROW REDUCE

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{array}{cccc} w_1 & w_2 & w_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad w = [2/5, 1/5, 2/5]$$

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Try it out:

$$[2/5, 1/5, 2/5] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = w = [2/5, 1/5, 2/5]$$

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

This does not mean, yet, that a stable vector has been found. The matrix P is regular since P^3 had no zero entries. By the theorem, P has a stable vector. By our earlier reasoning, there is only one vector with wP = w and it is the stable vector.

So the w we found is the stable vector.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

$$P^4 = \begin{bmatrix} .44 & .19 & .38 \\ .38 & .25 & .38 \\ .38 & .19 & .44 \end{bmatrix}$$

Two significant digits (notice rows don't add up to 1 due to round off error).

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

w = [.4, .2, .4]

$$P^8 = \begin{bmatrix} .40 & .20 & .40 \\ .40 & .20 & .40 \\ .40 & .20 & .40 \end{bmatrix}$$

Two significant digits. More significant digits woul reveal the result is not exact.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

w = [.4, .2, .4]

$$P^8 = \begin{bmatrix} .4023 & .1992 & .3984 \\ .3984 & .2031 & .3984 \\ .3984 & .1992 & .4023 \end{bmatrix}$$

Four significant digits (notice rows don't add up to 1 due to round off error).

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

w = [.4, .2, .4]

$$P^{16} = \begin{bmatrix} .4000 & .2000 & .4000 \\ .4000 & .2000 & .4000 \\ .4000 & .2000 & .4000 \end{bmatrix}$$

More than 4 significant digits would show that the rows are not exactly "w".

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

w = [2/5, 1/5, 2/5] $\quad zP^n \to w$ for all $z = [z_1, z_2, z_3]$.

Try z = [1, 0, 0]. Then zP^n = [1, 0, 0]P^n = first row of P^n.
So the first row of P^n should look more like w = [2/5, 1/5, 2/5] as n gets higher and higher.

w = [.4, .2, .4]

$$P^{16} = \begin{bmatrix} .4000 & .2000 & .4000 \\ .4000 & .2000 & .4000 \\ .4000 & .2000 & .4000 \end{bmatrix}$$

All rows converge to w.

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

w = [1/2, 1/2]

$$[1/2, 1/2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [1/2, 1/2]$$

33

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

w = [1/2, 1/2]

$$[1/2, 1/2] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [1/2, 1/2]$$

However, w is NOT a stable vector.

$$[1, 0] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [0, 1] \qquad [0, 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [1, 0]$$

$$[1, 0] \xrightarrow{P} [0, 1] \xrightarrow{P} [1, 0] \text{ etc.}$$ doesn't converge to w.

34

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

35

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} & \\ & \end{bmatrix} \end{array}$$

36

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} & 1/2 \\ & \end{bmatrix} \end{array}$$

37

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ & \end{bmatrix} \end{array}$$

38

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & \end{bmatrix} \end{array}$$

39

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

40

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

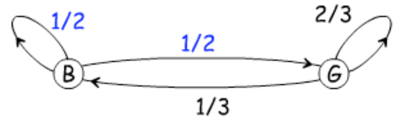
$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

41

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. What is the transition matrix?

Two states, bad and good, denoted B and G.

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$



42

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now?

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now?

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$(P^n)_{i,j} = P_{i,j}(n) =$ probability that the system is in state j after n transitions if it starts out in state i.

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

$$P = \begin{array}{c c} & \begin{array}{cc} B & G \end{array} \\ \begin{array}{c} B \\ G \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$(P^3)_{2,1} = P_{G,B}(3) =$ probability that the system is in state B after 3 transitions if it starts out in state G.

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^2 \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^2 \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 29/72 & 43/72 \\ 43/108 & 65/108 \end{bmatrix}$$

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}^2 \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 29/72 & 43/72 \\ \boxed{43/108} & 65/108 \end{bmatrix}$$

48

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

ERASE

$$\begin{array}{cc} & \begin{array}{cc} B & \end{array} \\ G & \begin{bmatrix} 29/72 & 43/72 \\ \boxed{43/108} & 65/108 \end{bmatrix} \end{array}$$

49

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. If the weather is good today, what is the probability that it will be bad three days from now? $P_{G,B}(3) = (P^3)_{2,1}$

ERASE

Notice that P^3 is also a transition matrix, since a transition applied 3 times is again a transition. The rows of P^n also add up to 1 for any n.

$$\begin{bmatrix} 29/72 & 43/72 \\ 43/108 & 65/108 \end{bmatrix}$$

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Lecture 31

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

Notice that P is a regular matrix since P^1 has no zero entries. So P has a stable vector.

51

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\boxed{(P^T - I)w^T = 0}$$

52

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\boxed{(P^T - I)w^T = 0}$$

$$PT - I = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/3 \\ 1/2 & -1/3 \end{bmatrix}$$

53

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 1/3 \\ 1/2 & -1/3 \end{bmatrix}$$

54

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 1/3 & 0 \\ 1/2 & -1/3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

55

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

ERASE

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 1/3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

56

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 1/3 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

57

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 1/3 & 0 \\ 0 & 5/3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

58

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 5/2 & -5/3 & 0 \\ 0 & 5/3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

59

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 5/2 & 0 & 1 \\ 0 & 5/3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

60

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

61

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$w = [2/5, 3/5]$

$$\begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

62

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$w = [2/5, 3/5]$

$$[2/5,\ 3/5] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [2/5,\ 3/5]$$

On average 2/5 of the days will have bad weather, 3/5 good.

63

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$w = [2/5, 3/5]$

$$[2/5,\ 3/5] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [2/5,\ 3/5]$$

On average 2/5 of the days will have bad weather, 3/5 good.

This is true no matter how things start out: $zP^n \to w$.

64

EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

w = [2/5, 3/5]

EXAMPLE: Start off getting a very accurate weather report that there is a 10% chance of bad weather of tomorrow. What happens in the long run?

On average 2/5 of the days will have bad weather, 3/5 good. This is true no matter how things start out: $zP^n \to w$.

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

w = [2/5, 3/5]

EXAMPLE: Start off getting a very accurate weather report that there is a 10% chance of bad weather of tomorrow. What happens in the long run? [.1, .9] → [.35, .65] → → [.3916666, .6083333] → [.39861111, .60138888] → → [.39976851, .60023148] → [.39996141, .60003858] → → [.3999935, .600006430] ≈ [2/5, 3/5]

Based on the prediction of .1 prob for bad weather tomorrow, the prediction for 7 days later is Pr[Bad weather] ≈ 2/5.

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EXAMPLE: On any given day the weather is in one of two states, good or bad. If the weather is bad, the probability that it will be good tomorrow is 1/2. If the weather is good, the probability that it will be bad tomorrow is 1/3. Find the stable vector for P.

$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

w = [2/5, 3/5]

Alternate view: If there are two countries B and G and the people move each year according to the transition matrix (e.g. exactly 1/3 of the people in Country B are selected *at random* to move to A). Then no matter how things start out, eventually (very close to) 2/5 of the people will be in country A and 3/5 in Country B, and it will stay close to that proportion.

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EXAMPLE: Find the stable vector for

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

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EXAMPLE: Find the stable vector for

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$[x, 1 - x]\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [x, 1 - x]$$

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EXAMPLE: Find the stable vector for

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$[x, 1 - x]\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [x, 1 - x]$$

$(1/2)x + (1/3)(1 - x) = x$

$(1/2)x - (1/3)x + 1/3 = x$

$(1/6)x + 1/3 = x$

$1/3 = (5/6)x$

70

EXAMPLE: Find the stable vector for

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$[x, 1 - x]\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [x, 1 - x]$$

$(1/2)x + (1/3)(1 - x) = x$

$(1/2)x - (1/3)x + 1/3 = x$

$(1/6)x + 1/3 = x$

$1/3 = (5/6)x$

$2/5 = x \qquad 3/5 = 1 - x$

[2/5, 3/5] = w

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EXAMPLE: Find the stable vector for the following transition matrix:

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

w = [c/(b + c), b/(b + c)]

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \qquad \frac{(1/3)}{(1/3 + 1/2)} = \frac{(2/6)}{(2/6 + 3/6)} = 2/5$$

w = [2/5, 3/5]

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EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$[x, y, 1-x-y]\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [x, y, 1-x-y]$$

$(1/2)x + 0y + (1/2)(1-x-y) = x$

$(1/2)x + 0y + 0(1-x-y) = y$

73

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$[x, y, 1-x-y]\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [x, y, 1-x-y]$$

$(1/2)x + 0y + (1/2)(1-x-y) = x$

$(1/2)x + 0y + 0(1-x-y) = y$

$-x - (1/2)y = -1/2$

$(1/2)x - y = 0$

74

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$[x, y, 1-x-y]\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [x, y, 1-x-y]$$

$(1/2)x + 0y + (1/2)(1-x-y) = x$

$(1/2)x + 0y + 0(1-x-y) = y$

$-x - (1/2)y = -1/2$

$(1/2)x - y = 0$

$x = 2/5 \quad y = 1/5$

$w = [2/5, 1/5, 2/5]$

75

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$[x, y, 1-x-y]\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [x, y, 1-x-y]$$

This was easier because there were two 0's in column 2.

$(1/2)x + 0y + (1/2)(1-x-y) = x$

$(1/2)x + 0y + 0(1-x-y) = y$

$-x - (1/2)y = -1/2$

$(1/2)x - y = 0$

$x = 2/5 \quad y = 1/5$

$w = [2/5, 1/5, 2/5]$

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EXAMPLE: Find the stable vector for

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \Rightarrow (P^T - I)w^T = \mathbf{0} \Rightarrow$

$PT - I = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/3 \\ 1/2 & -1/3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1/2 & 1/3 & 0 \\ 1/2 & -1/3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

RR $\Rightarrow \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$ $w = [2/5, 3/5]$ $[x, 1-x]\begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} = [x, 1-x]$

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$(P^n)_{i,j} = P_{i,j}(n) =$ probability that the system is in state j after n transitions if it starts out in state i.

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