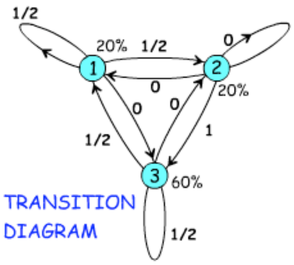
Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6



TRANSITION DIAGRAM

ON THE NEXT TRANSITION:

If the system is in state 1:

It has a probability of 1/2 ending up in state 2, and probability of 1/2 of staying in state 1.

If the system is in state 2:

It has a probability of 1 ending up in state 3.

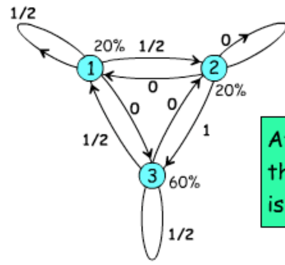
If the system is in state 3:

It has a probability of 1/2 ending up in state 1, and probability of 1/2 of staying in state 3.

1

Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6

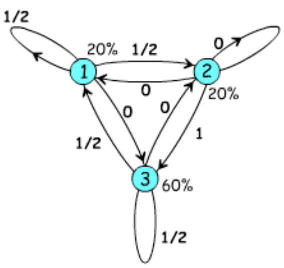


After one transition, what is the probabililty that the system is in state 1? 2? 3?

2

Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6

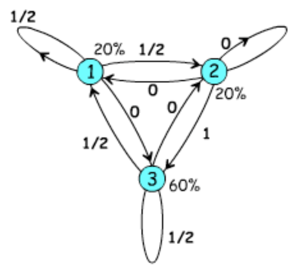


Draw a tree:

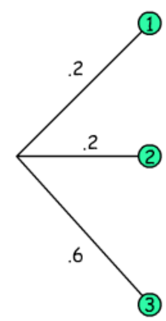
3

Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6



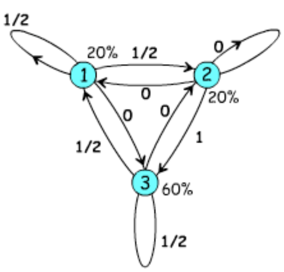
Draw a tree:



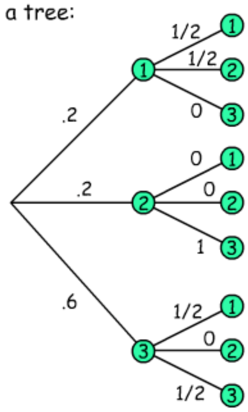
4

Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6



Draw a tree:



5

Initially,

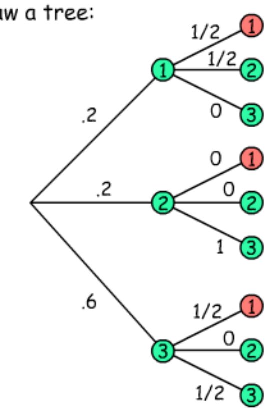
Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6

After 1 transition:

Pr[system in state 1] =

.2 · 1/2 + .2 · 0 + .6 · 1/2 = .4

Draw a tree:



6

Initially,

Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6

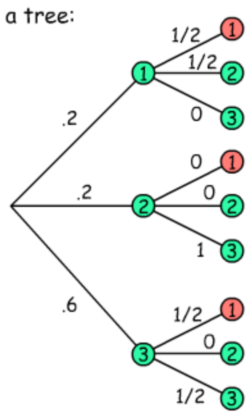
After 1 transition:

Pr[system in state 1] =

.2 · 1/2 + .2 · 0 + .6 · 1/2 = .4

$$= [.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = .4$$

Draw a tree:



7

Initially,

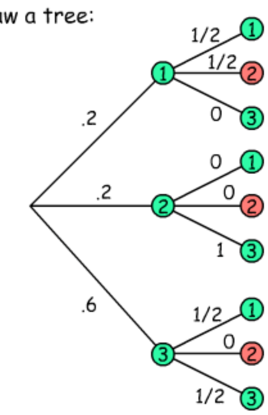
Pr[system in state 1] = .2
Pr[system in state 2] = .2
Pr[system in state 3] = .6

After 1 transition:

Pr[system in state 2] =

.2 · 1/2 + .2 · 0 + .6 · 0 = .1

Draw a tree:



8

Initially,

Pr[system in state 1] = .2

Pr[system in state 2] = .2

Pr[system in state 3] = .6

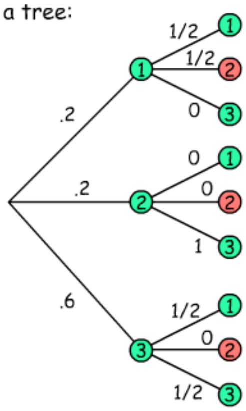
After 1 transition:

Pr[system in state 2] =

$.2 \cdot 1/2 + .2 \cdot 0 + .6 \cdot 0 = .1$

$$= [.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} = .1$$

Draw a tree:



9

Initially,

Pr[system in state 1] = .2

Pr[system in state 2] = .2

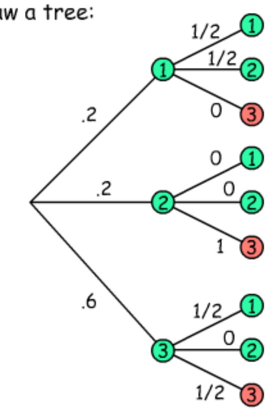
Pr[system in state 3] = .6

After 1 transition:

Pr[system in state 3] =

$.2 \cdot 0 + .2 \cdot 1 + .6 \cdot 1/2 = .5$

Draw a tree:



10

Lecture 30

Initially,

Pr[system in state 1] = .2

Pr[system in state 2] = .2

Pr[system in state 3] = .6

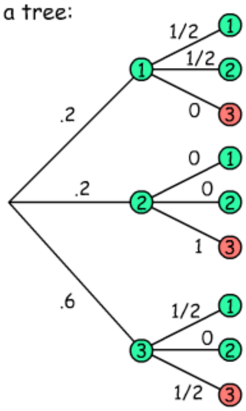
After 1 transition:

Pr[system in state 3] =

$.2 \cdot 0 + .2 \cdot 1 + .6 \cdot 1/2 = .5$

$$= [.2, .2, .6] \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = .5$$

Draw a tree:



11

Pr[state 1 after 1 trans.]:

$$[.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = .4$$

Pr[state 2 after 1 trans.]:

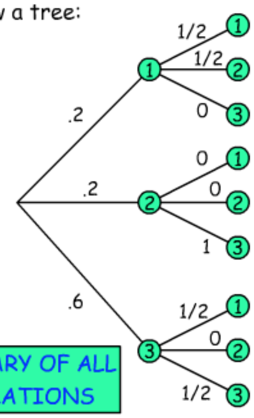
$$[.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix} = .1$$

Pr[state 3 after 1 trans.]:

$$[.2, .2, .6] \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = .5$$

SUMMARY OF ALL CALCULATIONS

Draw a tree:



12

$$[.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \quad .4$$

$$[.2, .2, .6] \quad \begin{matrix} 1/2 \\ 0 \\ 0 \end{matrix} \quad .1$$

$$[.2, .2, .6] \quad \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \quad .5$$

13

[.2, .2, .6]

[.2, .2, .6] [1/2 0 1/2] 1/2 0 0 [0 1 1/2]

.4

.1

.5

[.2, .2, .6]

14

Pr[state 1 after 1 trans.]:

$$[.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = .4$$

$$[.2, .2, .6] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [.4, .1, .5]$$

Pr[state 3 after 1 trans.]:

$$[.2, .2, .6] \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} = .5$$

15

Pr[state 1 after 1 trans.]:

$$[.2, .2, .6] \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = .4$$

$$[.2, .2, .6] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = [.4, .1, .5]$$

D_0 Initial prob. distribution

P Transition matrix

D_1 Prob. distribution after one transition

16

General Pattern: If the initial probability distribution is called D_0 (in this case it was [.2, .2, .6]) and if the transition matrix is called P then the probabilities for the new distribution D_1 are given by:

$$D_0 P = D_1$$

17

General Pattern: If the initial probability distribution is called D_0 (in this case it was [.2, .2, .6]) and if the transition matrix is called P then the probabilities for the new distribution D_1 are given by:

$$D_0 P = D_1$$

The distribution after 2 transitions is given by:

$$D_2 = D_1 P = (D_0 P)P = D_0 (P P) = D_0 P^2 = D_2$$

18

General Pattern: If the initial probability distribution is called D_0 (in this case it was [.2, .2, .6]) and if the transition matrix is called P then the probabilities for the new distribution D_1 are given by:

$$D_0 P = D_1$$

The distribution after 2 transitions is given by:

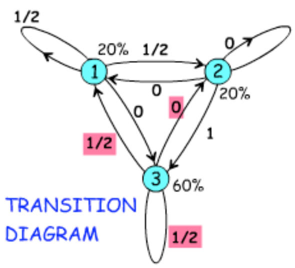
$$D_2 = D_1 P = (D_0 P)P = D_0 (P P) = D_0 P^2 = D_2$$

And in general:

$$D_0 P^n = D_n$$

19

COMMENT: It is possible to write down the transition matrix directly from the transition diagram (and visa-versa).



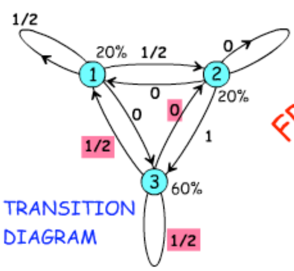
TRANSITION DIAGRAM

	1	2	3
1	1/2	1/2	0
2	0	0	1
3	1/2	0	1/2

Probabilities running "out of 3" into the other states

20

COMMENT: It is possible to write down the transition matrix directly from the transition diagram (and visa-versa).



TRANSITION DIAGRAM

TO

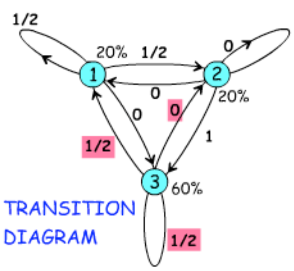
FROM

	1	2	3
1	1/2	1/2	0
2	0	0	1
3	1/2	0	1/2

Probabilities running "out of 3" into the other states

21

COMMENT: It is possible to write down the transition matrix directly from the transition diagram (and visa-versa).



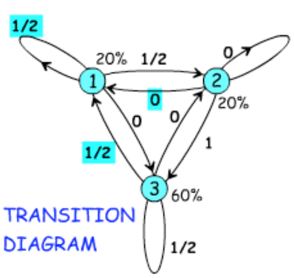
TRANSITION DIAGRAM

	1	2	3
1	1/2	1/2	0
2	0	0	1
3	1/2	0	1/2

Notice that the rows add up to 1 since they indicate the prob. of a transition to some (any) state.

22

COMMENT: It is possible to write down the transition matrix directly from the transition diagram (and visa-versa).



TRANSITION DIAGRAM

	1	2	3
1	1/2	1/2	0
2	0	0	1
3	1/2	0	1/2

Probabilities running "into 1" from the other states

23

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

24

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

To answer this, the initial probability distribution is

[1, 0, 0]

25

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

To answer this, the initial probability distribution is

[1, 0, 0]

Now multiply by P^3 to get the probability distribution after 3 transitions.

26

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

$$[1, 0, 0]\begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix}$$

Now multiply by P^3 to get the probability distribution after 3 transitions.

$$P = \begin{bmatrix}1/2 & 1/2 & 0\\ 0 & 0 & 1\\ 1/2 & 0 & 1/2\end{bmatrix} \quad P^2 = \begin{bmatrix}1/4 & 1/4 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/4 & 1/4\end{bmatrix} \quad P^3 = \begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix}$$

27

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

$$[1, 0, 0]\begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix} = [3/8, 1/8, 1/2]$$

Now multiply by P^3 to get the probability distribution after 3 transitions.

28

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

$$[1, 0, 0]\begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix} = [3/8, 1/8, 1/2]$$

Now multiply by P^3 to get the probability distribution after 3 transitions.

After 3 transitions,

Pr[System is in state 2 after 3 transitions] = 1/8

29

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

$$[1, 0, 0]\underbrace{\begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix}}_{P^3} = [3/8, 1/8, 1/2]$$

Notice: This is just the first row of P^3.

$(P^3)_{1,2}$ = probability that the system is in state 2 after 3 transitions if it starts out in state 1.

30

EXAMPLE: Suppose the system is in state 1. What is the probability that it will be in state 2 after 3 transitions?

$$[1, 0, 0]\underbrace{\begin{bmatrix}3/8 & 1/8 & 1/2\\ 1/2 & 1/4 & 1/4\\ 3/8 & 1/4 & 3/8\end{bmatrix}}_{P^3} = [3/8, 1/8, 1/2]$$

Notice: This is just the first row of P^3.

$(P^3)_{i,j}$ = probability that the system is in state j after 3 transitions if it starts out in state i.

31

$(P^n)_{i,j}$ = probability that the system is in state j after n transitions if it starts out in state i.

32

33

Notation:

$P_{i,j}(n)$ = probability that the system is in state j after n transitions if it starts out in state i.

$(P^n)_{i,j} = P_{i,j}(n)$ = probability that the system is in state j after n transitions if it starts out in state i.

34

$$P^n = \begin{bmatrix} P_{1,1}(n) & P_{1,2}(n) & P_{1,3}(n) \\ P_{2,1}(n) & P_{2,2}(n) & P_{2,3}(n) \\ P_{3,1}(n) & P_{3,2}(n) & P_{3,3}(n) \end{bmatrix}$$

$(P^n)_{i,j} = P_{i,j}(n)$ = probability that the system is in state j after n transitions if it starts out in state i.

35

$$P^n = \begin{bmatrix} P_{1,1}(n) & P_{1,2}(n) & P_{1,3}(n) \\ P_{2,1}(n) & P_{2,2}(n) & P_{2,3}(n) \\ P_{3,1}(n) & P_{3,2}(n) & P_{3,3}(n) \end{bmatrix}$$

$P_{2,3}(n)$: probability that after n transitions, the system will be in state 3 if it starts in state 2.

$(P^n)_{i,j} = P_{i,j}(n)$ = probability that the system is in state j after n transitions if it starts out in state i.

36

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

37

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 3/8 & 1/8 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 3/8 & 1/4 & 3/8 \end{bmatrix}$$

38

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 3/8 & 1/8 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 3/8 & 1/4 & 3/8 \end{bmatrix}$$

NO ZEROS ⇒ P REGULAR

39

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$$

regular?

Here * indicates an entry > 0. Notice that the product of 2 * terms is positive (i.e. *). So for example,

$$* \cdot * + * \cdot 0 + 0 \cdot 0 = * > 0$$

40

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix}$$

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$$

regular?

$$P^3 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

NO ZEROS ⇒ P REGULAR

41

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$$

regular?

$$P^3 = \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

NO ZEROS ⇒ P REGULAR

42

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

regular?

43

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

44

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

regular?

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = P^4 = P^6 = P^{2n} \qquad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P^5 = P^{2n+1}$$

45

Definition: A transition matrix, P, is called **regular** if for some n = 1, 2, . . . the matrix P^n has NO zero entries.

Example: Is the matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

P is not regular

regular?

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = P^4 = P^6 = P^{2n} \qquad P^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P^5 = P^{2n+1}$$

46

Definition: A vector $w = [w_1, w_2, \dots, w_n]$ is called a probability vector if

1) $w_1 + w_2 + \dots + w_n = 1$.

2) $w_i \geq 0$ for $i = 1, 2, \dots, n$.

47

Definition: A vector $w = [w_1, w_2, \dots, w_n]$ is called a probability vector if

1) $w_1 + w_2 + \dots + w_n = 1$.

2) $w_i \geq 0$ for $i = 1, 2, \dots, n$.

In the context of Markov chains, these are sometimes called state vectors.

48

Definition: A vector $w = [w_1, w_2, \ldots, w_n]$ is called a probability vector if

1) $w_1 + w_2 + \ldots + w_n = 1.$

2) $w_i \geq 0$ for $i = 1, 2, \ldots, n$.

In the context of Markov chains, these are sometimes called state vectors.

Example: [.2, .2, .6] In general we'll use these to represent the probabilities of a system to be in different states. In this case Pr[System in state 1] = .2, Pr[State 2] = .2, Pr[State 3] = .6.

Definition: A Markov chain is said to have a stable vector $w = [w_1, w_2, \ldots, w_n]$ if w is a state vector and

1) $wP = w$ (P = transition matrix for the chain)

2) Given any state vector $z = [z_1, z_2, \ldots, z_n]$,

$$zP^n \to w$$

This last line means that the entries of the vector zP^n get closer and closer (eventually) to the entries of w as n increases.

Definition: A Markov chain is said to have a stable vector $w = [w_1, w_2, \ldots, w_n]$ if w is a state vector and

1) $wP = w$ (P = transition matrix for the chain)

2) Given any state vector $z = [z_1, z_2, \ldots, z_n]$,

$$zP^n \to w$$

Comment: If there is a stable vector, it is the only state vector with $wP = w$. If $qP = q$ and q is a state vector, then $qP^n \to w$ since w is stable. But $qP^n = q$, so $q \to w$, which implies q = p.

Theorem: Every regular Markov chain has a stable vector.

There is a state vector w such that $wP = w$, and it is the only vector with this property.

Furthermore, given any state vector z, $zP^n \to w$.

THE TRANSPOSE OF A MATRIX:

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Here is a 3 x 2 matrix. It has a transpose called A^T.

$$A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

A^T is obtained by interchanging rows and columns - first row of A is the first column of A^T, second row of A is the second column of A^T, etc.

THE TRANSPOSE OF A MATRIX:

Fact: $(AB)^T = B^T A^T$

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Basic example:

$$[1, 2, 3, 4] \begin{bmatrix} 1 \\ 4 \\ 1 \\ 6 \end{bmatrix} = [36] \qquad [1, 4, 1, 6] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [36]$$

$$1 \cdot 1 + 2 \cdot 4 + 3 \cdot 1 + \underline{4 \cdot 6} = 36 = 1 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 + \underline{6 \cdot 4}$$

order of mult. reversed

THE TRANSPOSE OF A MATRIX:

Fact: $(AB)^T = B^T A^T$

$$\left(\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} r & s & t & u \\ v & w & x & y \end{bmatrix} \right)^T \overset{?}{=} \begin{bmatrix} r & v \\ s & w \\ t & x \\ u & y \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

3,2 entry: Calculate 2,3 entry of the product. This will be the 3,2 entry of the transpose of the product.

ct + dx

THE TRANSPOSE OF A MATRIX:

Fact: $(AB)^T = B^T A^T$

$$\left[\begin{bmatrix} a & b\\ c & d\\ e & f\end{bmatrix}\begin{bmatrix} r & s & t & u\\ v & w & x & y\end{bmatrix}\right]^T \stackrel{?}{=} \begin{bmatrix} r & v\\ s & w\\ t & x\\ u & y\end{bmatrix}\begin{bmatrix} a & c & e\\ b & d & f\end{bmatrix}$$

3,2 entry:

ct + dx

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THE TRANSPOSE OF A MATRIX:

Fact: $(AB)^T = B^T A^T$

$$\left[\begin{bmatrix} a & b\\ c & d\\ e & f\end{bmatrix}\begin{bmatrix} r & s & t & u\\ v & w & x & y\end{bmatrix}\right]^T \stackrel{?}{=} \begin{bmatrix} r & v\\ s & w\\ t & x\\ u & y\end{bmatrix}\begin{bmatrix} a & c & e\\ b & d & f\end{bmatrix}$$

3,2 entry:

ct + dx

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FINDING A STABLE VECTOR:

$wP = w$

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FINDING A STABLE VECTOR:

$wP = w$

$(wP)^T = w^T$

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FINDING A STABLE VECTOR:

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$P^T w^T = w^T$

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FINDING A STABLE VECTOR:

$wP = w$

$(wP)^T = w^T$

$P^T w^T = w^T$

$P^T w^T - w^T = \mathbf{0}$ Here **0** is vector. It looks like:

$$\begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}$$ if w has three entries.

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FINDING A STABLE VECTOR:

$wP = w$

$(wP)^T = w^T$

$P^T w^T = w^T$

$P^T w^T - w^T = \mathbf{0}$

$P^T w^T - I w^T = \mathbf{0}$

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FINDING A STABLE VECTOR:

$wP = w$

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$(P^T - I) w^T = \mathbf{0}$

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FINDING A STABLE VECTOR:

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FINDING A STABLE VECTOR:

$(P^T - I)w^T = \mathbf{0}$

This has to be solved for w (or w^T).

The technique for this is already developed.

Solve: $Ax = b$ (in this case $b = \mathbf{0}$)

Form the augmented matrix

$[A|b]$

and row reduce.

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Lecture 30

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$

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$$P^T - I = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix}$$

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$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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THIS CORRESPONDS TO A SYSTEM OF 3 EQUATIONS AND 3 UNKNOWNS. EXAMPLE:

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$(P^T - I)w^T = \mathbf{0}$ $\quad w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -1/2\, w_1 + 0w_2 + 1/2\, w_3 = 0$$

THIS CORRESPONDS TO A SYSTEM OF 3 EQUATIONS AND 3 UNKNOWNS.

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$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -1/2\, w_1 + 0w_2 + 1/2\, w_3 = 0$$

THERE IS ONE MORE EQUATION NOT SHOWN HERE

$w_1 + w_2 + w_3 = 1$ SINCE w IS A STATE VECTOR.

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EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$w_1 + w_2 + w_3 = 1$ SINCE w IS A STATE VECTOR.

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Lecture 30

EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $w = [w_1, w_2, w_3]$

$$\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & -1/2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

FORM THE AUGMENTED MATRIX

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EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $w = [w_1, w_2, w_3]$

$$\begin{matrix} w_1 & w_2 & w_3 & \\ \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

NOW ROW REDUCE

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$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$(P^T - I)w^T = \mathbf{0}$ $w = [w_1, w_2, w_3]$

$$\begin{matrix} w_1 & w_2 & w_3 & \\ \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad w = [2/5, 1/5, 2/5]$$

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EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Try it out:

$$[2/5, 1/5, 2/5] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix} = w = [2/5, 1/5, 2/5]$$

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EXAMPLE: Find a stable vector, if it exists, for

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

This does not mean, yet, that a stable vector has been found. The matrix P is regular since P^3 had no zero entries. By the theorem, P has a stable vector. By our earlier reasoning, there is only one vector with wP = w and it is the stable vector.

So the w we found is the stable vector.

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