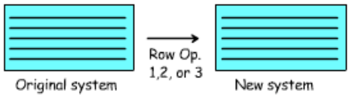
General Picture:



1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another.

ANY SOLUTION TO THE ORIGINAL SYSTEM IS A SOLUTION TO THE NEW SYSTEM.

ANY SOLUTION TO THE NEW SYSTEM IS A SOLUTION TO THE ORIGINAL SYSTEM. WHY???

I TOLD YOU SO.

EXAMPLE: Solve the system

$$\begin{aligned} 1x_1 + 2x_2 + 0x_3 &= 19 \\ -1x_1 + 1x_2 + 1x_3 &= 10 \\ -2x_1 + 0x_2 + 2x_3 &= 6 \end{aligned}$$

EXAMPLE: Solve the system

$$\begin{bmatrix} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{bmatrix}$$

Recall that we can always put the x_i and the + and = signs back in whenever needed.

EXAMPLE: Solve the system

$$\begin{bmatrix} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{bmatrix} \xrightarrow[\text{REDUCE}]{\text{ROW}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

EXAMPLE: Solve the system

$$\begin{bmatrix} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{bmatrix} \xrightarrow[\text{REDUCE}]{\text{ROW}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

↓ Reconstitute

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 5 \\ 0x_1 + 1x_2 + 0x_3 &= 7 \\ 0x_1 + 0x_2 + 1x_3 &= 8 \end{aligned}$$

EXAMPLE: Solve the system

$$\begin{bmatrix} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{bmatrix} \xrightarrow[\text{REDUCE}]{\text{ROW}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix}$$

↓ Reconstitute

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 5 \\ 0x_1 + 1x_2 + 0x_3 &= 7 \\ 0x_1 + 0x_2 + 1x_3 &= 8 \end{aligned}$$

←

$$x_1 = 5, \quad x_2 = 7, \quad x_3 = 8$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

IN GENERAL YOU CAN'T ALWAYS GET A NICE "DIAGAONAL" MATRIX ON THE LEFT. SO YOU AIM FOR SOMETHING CALLED "REDUCED FORM."

DEFINITION:

A matrix is said to be in reduced form if

1) All "zero rows" appear at the bottom of the matrix.
2) The lead nonzero entry in every row is a 1. These lead nonzero entries are called pivots.
3) ALL entries above and below a pivot are 0.
4) If j > i and if Row i and Row j both have nonzero entries, then the lead nonzero entry of Row j appears to the right of the lead nonzero entry of Row i .

EXAMPLES:

$$\begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & 15 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & 15 & 3 & 0 & 0 & 12 \\ 0 & 0 & 1 & 3 & 5 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 43 \end{bmatrix}$$

THIS IS WHAT YOU ALWAYS AIM FOR

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2\\ 2x_1 + 3x_2 - 3x_3 &= 8\\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

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USE ROW OPERATIONS:

1) Interchange rows.

2) Multiply a row by a nonzero constant.

3) Add a multiple of one row to another.

AUGMENTED MATRIX:

$$\begin{bmatrix}0&1&1&2\\2&3&-3&8\\1&2&-1&5\end{bmatrix}$$

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AUGMENTED MATRIX:

$$\begin{bmatrix}0&1&1&2\\2&3&-3&8\\1&2&-1&5\end{bmatrix} \to \begin{bmatrix}1&0&-3&1\\0&1&1&2\\0&0&0&0\end{bmatrix}$$

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$$\begin{aligned} x_2 + x_3 &= 2\\ 2x_1 + 3x_2 - 3x_3 &= 8\\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

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$$\begin{matrix}x_1 & x_2 & x_3 & \\ 1&0&-3&1\\0&1&1&2\\0&0&0&0\end{matrix}$$

1) Circle pivots.
2) Remaining columns yield free variables.
3) Solve for the other variables in terms of the free variables. This always works.

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$$\begin{aligned} x_2 + x_3 &= 2\\ 2x_1 + 3x_2 - 3x_3 &= 8\\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

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$$\begin{bmatrix}0&1&1&2\\2&3&-3&8\\1&2&-1&5\end{bmatrix} \to \begin{bmatrix}1&0&-3&1\\0&1&1&2\\0&0&0&0\end{bmatrix}$$

$$\begin{matrix}x_1 & x_2 & x_3 & \\ 1&0&-3&1\\0&1&1&2\\0&0&0&0\end{matrix}$$

1) Circle pivots. The lead nonzero entries in each row (arranged = 1).

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1) Circle pivots.
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RECONSTITUTE:

$$\begin{aligned} 1x_1 + 0x_2 - 3x_3 &= 1\\ 0x_1 + 1x_2 + 1x_3 &= 2\\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

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1) Circle pivots.
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RECONSTITUTE:

$$\begin{aligned} 1x_1 + 0x_2 - 3x_3 &= 1\\ 0x_1 + 1x_2 + 1x_3 &= 2 \end{aligned}$$

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3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

USE ROW OPERATIONS:

1) Interchange rows.
2) Multiply a row by a nonzero constant.
3) Add a multiple of one row to another.

AUGMENTED MATRIX:

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & -3 & 8 \\ 1 & 2 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & \\ \end{matrix}$$
$$\begin{bmatrix} (1) & 0 & -3 & 1 \\ 0 & (1) & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Circle pivots.
2) Remaining columns yield free variables.
3) Solve for the other variables in terms of the free variables. This always works.

RECONSTITUTE:

$$1\,x_1 + 0\,x_2 - 3\,x_3 = 1$$
$$0\,x_1 + 1\,x_2 + 1\,x_3 = 2$$

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

USE ROW OPERATIONS:

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RECONSTITUTE:

$$1\,x_1 - 3\,x_3 = 1$$
$$1\,x_2 + 1\,x_3 = 2$$

3 EQS. & 3 UNKNOWNS:

Example:

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1) Circle pivots.
2) Remaining columns yield free variables.
3) Solve for the other variables in terms of the free variables. This always works.

RECONSTITUTE:

$$1\,x_1 - 3\,x_3 = 1$$
$$1\,x_2 + 1\,x_3 = 2$$

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

If $x_3 = 12$,

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

If $x_3 = 12$, then one solution is:

$x_1 = 1 + 3x12 = 37$
$x_2 = 2 - 12 = -10$
$x_3 = 12$

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

For every value of x_3 there is one solution. There are an infinite number of solutions, generated by the one free parameter.

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE

COMMENT:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \quad \longleftarrow \text{This is the equation of a plane.} \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

$x_2 + x_3 = 2$ — The equation of a line in the $x_2\,x_3$ plane. Then x_1 free. This sweeps out a plane as x_1 varies.

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE

COMMENT:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \quad \longleftarrow \text{This is the equation of a plane.} \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

Each of these equations is the equation of a plane. Solving the system is equivalent to find the intersection of 3 planes, just as solving a system of 2 equations and 2 unknowns is equivalent to finding the intersection of two lines.

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE

COMMENT:

Example:

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Each of these equations is the equation of a plane. Solving the system is equivalent to find the intersection of 3 planes, just as solving a system of 2 equations and 2 unknowns is equivalent to finding the intersection of two lines. How could the intersection of 3 planes have an infinite number of points described by 1 free parameter????

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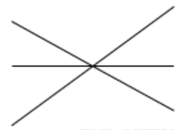
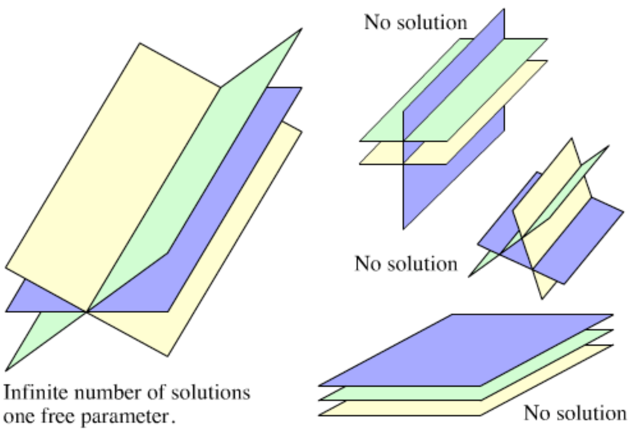


END VIEW

$x_1 = 1 + 3x_3$
$x_2 = 2 - x_3$
x_3 IS FREE



Infinite number of solutions
one free parameter.

INCONSISTENT SYSTEMS:

$$\begin{bmatrix} 0 & 1 & 1 & 3 \\ 2 & 3 & -3 & 8 \\ 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Last Line:

$$0x_1 + 0x_2 + 0x_3 = 1.$$

NOT POSSIBLE

This system has no solution.

It is INCONSISTENT.

EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

ASSOCIATED AUGMENTED MATRIX:

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 2 & -2 & 2 & -2 & 4 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

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$-2R_1 + R_2$:

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

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$(1/2)\ R_2$:

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

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$-1R_2 + R_3$:

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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EXAMPLE:

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The variables x_2 and x_4 variables are free. Solve for the other variables in terms of these.

$x_3 + x_4 = -1 \Rightarrow x_3 = -1 - x_4$

$x_1 - x_2 - 2x_4 = 3 \Rightarrow x_1 = 3 + x_2 + 2x_4$

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$x_1 - x_2 - 2x_4 = 3 \Rightarrow x_1 = 3 + x_2 + 2x_4$

$$\begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are free to be any real numbers.

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

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$x_3 + x_4 = -1 \Rightarrow x_3 = -1 - x_4$

$x_1 - x_2 - 2x_4 = 3 \Rightarrow x_1 = 3 + x_2 + 2x_4$

$$\begin{matrix} x_1 \to \\ x_2 \to \\ x_3 \to \\ x_4 \to \end{matrix} \begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are free to be any real numbers.

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

Example: $x_2 = 10$ and $x_4 = 20$ yields the solution

$$\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 10 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 20 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 53 \\ 10 \\ -21 \\ 20 \end{bmatrix}$$

$-1R_2 + R_3$:

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_2 and x_4 variables are free. Solve for the other variables in terms of these.

$x_3 + x_4 = -1 \Rightarrow x_3 = -1 - x_4$

$x_1 - x_2 - 2x_4 = 3 \Rightarrow x_1 = 3 + x_2 + 2x_4$

$$\begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are free to be any real numbers.

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

$-1R_2 + R_3$:

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_2 and x_4 variables are free. Solve for the other variables in terms of these.

$x_3 + x_4 = -1 \Rightarrow x_3 = -1 - x_4$

$x_1 - x_2 - 2x_4 = 3 \Rightarrow x_1 = 3 + x_2 + 2x_4$

In general, this describes a plane. There are an infinite number of solutions. described by two free parameters. →

$$\begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

where x_2 and x_4 are free to be any real numbers.

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

$-1R_2 + R_3$:

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix}$$
$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 pivots, 2 free parameters. Intersection is a plane.

What else could have happened??

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Inconsistent system, no solutions.

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

3 pivots, 1 free parameter. Intersection is a line.

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1 pivot, 3 free parameters. Intersection is a solid i.e. 3-space.

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EXAMPLE:

$$\begin{aligned} 1x_1 - 1x_2 + 0x_3 - 2x_4 &= 3 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 &= 4 \\ 1x_3 + 1x_4 &= -1 \end{aligned}$$

$-1R_2 + R_3$:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & -1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

2 pivots, 2 free parameters. Intersection is a plane.

Also recall that if you have 3 equations and 3 unknowns, then the solution can consist of one point only. This also occurs with n equations and n unknowns - but not always!!!

0) No solutions
1) One solution
2) An infinite number of solutions with 1 free parameter.
3) An infinite number of solutions with 2 free parameters.

ETC. Depending on the number of variables.

ROW REDUCTION ALGORITHM:

1) Go to the first NON-ZERO column. By interchanging rows, get the "top" entry to be nonzero.

2) By using row operations of type 3 (& type 2 if convenient), set all entries below that top entry to zero.

3) Now look at rows 2, 3, . . . and go to the first nonzero column. By interchanging rows get the top entry to be nonzero.

4) By applications of row operations of type 3) &2), get all zeroes below that nonzero entry.

5) Now look at row 3. Then look and row 4, etc. Repeat the process until all columns have been worked on.

6) The pivots are the lead nonzero entries in each row. By row operations of type 3, get all entries above the pivots to be zero.

7) By row operations of type 2, set all pivots to 1.

Lecture 19

DEFINITION:

A matrix is an m x n array of numbers.

Example: 3 x 4 (#Rows x #Columns)

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ \pi & 1 & \frac{1}{2} & -8 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

The (2,3) entry is $\frac{1}{2}$.

The (i,j) entry is the entry in the i^{th} row and j^{th} column.

NOTATION: If A is a matrix, the (a_{ij}) will be used sometimes to denote this matrix.

NOTATION: If A is a matrix, then (a will be used sometimes to denote this matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & a_{mn} \end{bmatrix}$$

EXAMPLES:

Column vectors:

$$\begin{bmatrix} 6 \\ -2 \\ 1/45 \\ 10 \end{bmatrix} \quad 4 \times 1$$

Row vectors:

$$\begin{bmatrix} 5 & 7 & -3 & 1.02 & 0 \end{bmatrix} \quad 1 \times 5$$

General matrices:

$$\begin{bmatrix} 2 & 7 & 1 & 9 \\ -8 & 95 & 1/3 & -50 \end{bmatrix} \quad 2 \times 4$$

ADDITION:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 1 & 1 & 1 \\ 2 & -2 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 6 \\ 3 & 7 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 10 \\ 3 & 11 & 12 \\ 1 & 1 & 1 \\ 5 & 1 & 8 \end{bmatrix}$$

SCALAR MULTIPLICATION:

$$3 \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 1 & 1 & 1 \\ 2 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 15 \\ 0 & 12 & 18 \\ 3 & 3 & 3 \\ 6 & -6 & 21 \end{bmatrix}$$

$$\begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 \cdot x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

PROPERTIES:

$A + B = B + A$
$cA + dA = (c + d) A$
$cA + cB = c(A + B)$
$(cd)A = c(dA)$

$$\begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 0 \\ -1x_4 \\ 1x_4 \end{bmatrix} = \begin{bmatrix} 3 + x_2 + 2x_4 \\ x_2 \\ -1 \cdot x_4 \\ x_4 \end{bmatrix}$$

A matrix times a column vector:

Example:

Call ↓ this matrix E:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 8 \\ 2 \end{bmatrix} \qquad 1 \cdot 1 + 2 \cdot 0 + 4 \cdot 1 = 5$$

4 x 3 3 x 1 4 x 1

A matrix times a column vector:

Example:

Call ↓ this matrix E:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 8 \\ 2 \end{bmatrix}$$

4 x 3 3 x 1 4 x 1

Notice:

$$\underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} \underset{c^1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \underset{E_{\cdot 1}}{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}}$$

A matrix times a column vector:

Example:

Call ↓ this matrix E:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 8 \\ 2 \end{bmatrix}$$

4 x 3   3 x 1   4 x 1

Notice:

$$\underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} \underset{c^1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \underset{E_{\bullet 1}}{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}}$$

$$\underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} \underset{c^2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \underset{E_{\bullet 2}}{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} \underset{c^3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \underset{E_{\bullet 3}}{\begin{bmatrix} 4 \\ 7 \\ 6 \\ 1 \end{bmatrix}}$$

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A row vector times a matrix:

Example:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \end{bmatrix} \underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 8 & 7 & 19 \end{bmatrix}$$

1 x 4   4 x 3   1 x 3

Notice:

$$\underset{r_1}{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{E_{1\bullet}}{\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}}$$

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$$\underset{r_2}{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{E_{2\bullet}}{\begin{bmatrix} 3 & 1 & 7 \end{bmatrix}}$$

$$\underset{r_3}{\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{E_{3\bullet}}{\begin{bmatrix} 2 & 1 & 6 \end{bmatrix}}$$

$$\underset{r_4}{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{E_{4\bullet}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}$$

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$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 4 & 8 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

2 x 3   3 x 2   2 x 2

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$$

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