System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned}$$

1

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned}$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

2

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{}$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

3

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{} \quad \begin{aligned} -2x_1 - 4x_2 + 2x_3 &= -10 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \end{aligned}$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

4

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{} \quad \begin{aligned} -2x_1 - 4x_2 + 2x_3 &= -10 \\ \underline{2x_1 + 3x_2 - 3x_3} &\underline{= 8} \\ 0x_1 - 1x_2 - 1x_3 &= -2 \end{aligned}$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

5

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{} \quad 0x_1 - 1x_2 - 1x_3 = -2$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

6

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{} \quad 0x_1 - 1x_2 - 1x_3 = -2$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system.

7

System of linear equations:

$$\begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned} \quad \xrightarrow[-2R_3+R_2]{} \quad \begin{aligned} 0x_1 + 1x_2 + 1x_3 &= 2 \\ 0x_1 - 1x_2 - 1x_3 &= -2 \\ 1x_1 + 2x_2 - 1x_3 &= 5 \end{aligned}$$

Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number ($\neq 0$).
3) Add a multiple of one equation to another

to get a new system. ANY SOLUTION TO THE ORIGINAL SYSTEM IS A SOLUTION TO THE NEW SYSTEM.

8

System of linear equations:

$$\begin{array}{l} 0x_1 + 1x_2 + 1x_3 = 2 \\ 2x_1 + 3x_2 - 3x_3 = 8 \\ 1x_1 + 2x_2 - 1x_3 = 5 \end{array} \xrightarrow{-2R_3+R_2} \begin{array}{l} 0x_1 + 1x_2 + 1x_3 = 2 \\ 0x_1 - 1x_2 - 1x_3 = -2 \\ 1x_1 + 2x_2 - 1x_3 = 5 \end{array}$$

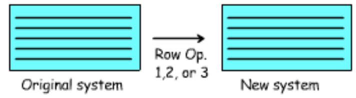
Apply one of the following "row" operations:

1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another

to get a new system. ANY SOLUTION TO THE ORIGINAL SYSTEM IS A SOLUTION TO THE NEW SYSTEM.

9

General Picture:

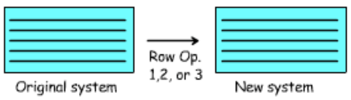


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

ANY SOLUTION TO THE ORIGINAL SYSTEM IS A SOLUTION TO THE NEW SYSTEM.

10

General Picture:



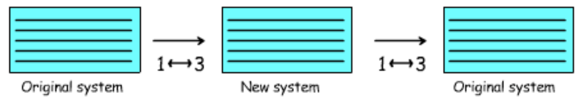
1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

ANY SOLUTION TO THE ORIGINAL SYSTEM IS A SOLUTION TO THE NEW SYSTEM.

ANY SOLUTION TO THE NEW SYSTEM IS A SOLUTION TO THE ORIGINAL SYSTEM. WHY???

11

General Picture:

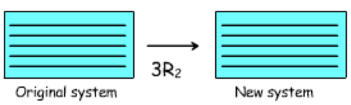


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

EXAMPLE: Suppose the row operation is of type 1) - interchange of rows 1 and 3. Then apply the same row operation again. This will get you back to the original system. So any solution of the new system is a solution of the original system.

12

General Picture:

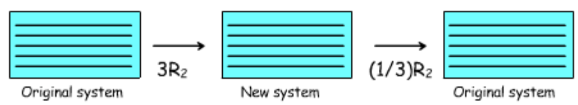


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

EXAMPLE: Suppose the row operation is of type 2) 3 x Row 2.

13

General Picture:

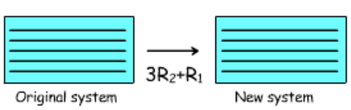


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

EXAMPLE: Suppose the row operation is of type 2) 3 x Row 2. Then apply another operation of type, namely 1/3 x Row 2. This will get you back to the original system. 1/3 x 3 x Row2 = Row2. So any solution of the new system is a solution of the original system, since solutions "pre row op" are solutions "post row op". Notice we are using a factor ≠ 0.

14

General Picture:

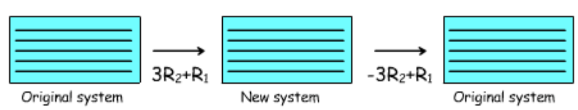


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

EXAMPLE: Suppose the row operation is of type 3) 3 x Row 2 added to Row 1.

15

General Picture:

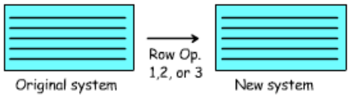


1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

EXAMPLE: Suppose the row operation is of type 3): 3 x Row 2 added to Row 1. Now take -3 x Row 2 added to Row 1. Overall, we have added 3 x Row 2 to Row 1 and then added -3 x Row 2 to Row 1. These cancel each other out, leaving Row 1 unchanged. But any solution to the new system is a solution after $-3R_2+R_1$ is performed. So any solution of the new system is a solution to the old system.

16

CONCLUSION:



1) Interchange 2 rows (i.e. equations).
2) Multiply one equation by a number (≠ 0).
3) Add a multiple of one equation to another.

THESE TWO SYSTEMS HAVE EXACTLY THE SAME SOLUTIONS

EXAMPLE: Solve the system

$$\begin{aligned} 1x_1 + 2x_2 + 0x_3 &= 19 \\ -1x_1 + 1x_2 + 1x_3 &= 10 \\ -2x_1 + 0x_2 + 2x_3 &= 6 \end{aligned}$$

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array}$$

Recall that we can always put the x_i and the + and = signs back in whenever needed.

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \qquad \begin{array}{c} \xrightarrow{2R_1} \\ \\ \xrightarrow{R_3} \end{array} \qquad \begin{array}{rrrr} 2 & 4 & 0 & 38 \\ & & & \\ -2 & 0 & 2 & 6 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \qquad \begin{array}{c} \xrightarrow{2R_1} \\ \\ \xrightarrow{R_3} \end{array} \qquad \begin{array}{rrrr} 2 & 4 & 0 & 38 \\ & & & \\ -2 & 0 & 2 & 6 \\ \hline 0 & 4 & 2 & 44 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \qquad \begin{array}{rrrr} & & & \\ & & & \\ & & & \\ 0 & 4 & 2 & 44 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \qquad \begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ & & & \\ 0 & 4 & 2 & 44 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \qquad \begin{array}{rrrr} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \xrightarrow{2R_1+R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

25

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \xrightarrow{2R_1+R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{R_1+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 3 & 1 & 29 \\ 0 & 4 & 2 & 44 \end{array}$$

26

Lecture 17

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \xrightarrow{2R_1+R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{R_1+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 3 & 1 & 29 \\ 0 & 4 & 2 & 44 \end{array} \xrightarrow{4R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 12 & 4 & 116 \\ 0 & 4 & 2 & 44 \end{array}$$

27

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \xrightarrow{2R_1+R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{R_1+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 3 & 1 & 29 \\ 0 & 4 & 2 & 44 \end{array} \xrightarrow{4R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 12 & 4 & 116 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{-3R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 0 & -2 & -16 \\ 0 & 4 & 2 & 44 \end{array}$$

28

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ -2 & 0 & 2 & 6 \end{array} \xrightarrow{2R_1+R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ -1 & 1 & 1 & 10 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{R_1+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 3 & 1 & 29 \\ 0 & 4 & 2 & 44 \end{array} \xrightarrow{4R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 12 & 4 & 116 \\ 0 & 4 & 2 & 44 \end{array}$$

$$\xrightarrow{-3R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 0 & -2 & -16 \\ 0 & 4 & 2 & 44 \end{array} \xrightarrow{R_2 \leftrightarrow R_3} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array}$$

29

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array}$$

30

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array}$$

31

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array}$$

32

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{2R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 0 & 28 \\ 0 & 0 & -2 & -16 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{2R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 0 & 28 \\ 0 & 0 & -2 & -16 \end{array}$$

$$\xrightarrow{(1/4)R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & -16 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{2R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 0 & 28 \\ 0 & 0 & -2 & -16 \end{array}$$

$$\xrightarrow{(1/4)R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{-2R_2+R_1} \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & -16 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 2 & 44 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{2R_3+R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 4 & 0 & 28 \\ 0 & 0 & -2 & -16 \end{array}$$

$$\xrightarrow{(1/4)R_2} \begin{array}{cccc} 1 & 2 & 0 & 19 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & -16 \end{array} \xrightarrow{-2R_2+R_1} \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & -16 \end{array}$$

$$\xrightarrow{(-1/2)R_3} \begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 8 \end{array}$$

EXAMPLE: Solve the system

$$\begin{array}{l} 1x_1 + 0x_2 + 0x_3 = 5 \\ 0x_1 + 1x_2 + 0x_3 = 7 \\ 0x_1 + 0x_2 + 1x_3 = 8 \end{array}$$

Reconstitute

EXAMPLE: Solve the system

$$\begin{array}{l} 1x_1 + 0x_2 + 0x_3 = 5 \\ 0x_1 + 1x_2 + 0x_3 = 7 \\ 0x_1 + 0x_2 + 1x_3 = 8 \end{array}$$

Reconstitute

$$x_1 = 5, \quad x_2 = 7, \quad x_3 = 8$$

Try it out:

$$\begin{array}{l} 1x_1 + 2x_2 + 0x_3 = 19 \\ -1x_1 + 1x_2 + 1x_3 = 10 \\ -2x_1 + 0x_2 + 2x_3 = 6 \end{array}$$

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

USE ROW OPERATIONS:

1) Interchange rows.

2) Multiply a row by a nonzero constant.

3) Add a multiple of one row to another.

AUGMENTED MATRIX:

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & -3 & 8 \\ 1 & 2 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 EQS. & 3 UNKNOWNS:

Example:

$$\begin{aligned} x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - 3x_3 &= 8 \\ x_1 + 2x_2 - x_3 &= 5 \end{aligned}$$

USE ROW OPERATIONS:

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AUGMENTED MATRIX:

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$$\begin{matrix} x_1 & x_2 & x_3 & \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Circle pivots.
2) Remaining columns yield free variables.
3) Solve for the other variables in terms of the free variables. This always works.

$$\begin{matrix} x_1 & x_2 & x_3 & \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Circle pivots. The lead nonzero entries in each row (arranged = 1).

$$\begin{matrix} x_1 & x_2 & x_3 & \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Circle pivots.
2) Remaining columns yield free variables.

$$\begin{matrix} x_1 & x_2 & x_3 & \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Circle pivots.
2) Remaining columns yield free variables.
3) Solve for the other variables in terms of the free variables. This always works.

$x_1 = 1 + 3x_3$

$x_2 = 2 - x_3$

x_3 is free to be anything

$\{(1 + 3x_3, 2 - x_3, x_3) : x_3 \in R\}$

DEFINITION:

A matrix is said to be in reduced form if

1) All "zero rows" appear at the bottom of the matrix.

2) The lead nonzero entry in every row is a 1. These lead nonzero entries are called pivots.

3) ALL entries above and below a pivot are 0.

4) If j > i and if Row i and Row j both have nonzero entries, then the lead nonzero

EXAMPLES:

$$\begin{bmatrix} 1 & 0 & 0 & 15 \\ 0 & 1 & 1 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & 15 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & 15 & 3 & 0 & 0 & 12 \\ 0 & 0 & 1 & 3 & 5 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 43 \end{bmatrix}$$