

BERNOULLI PROCESSES:

STOCHASTIC PROCESS: A stochastic process is a process in which one thing happens after another. Example: Anything involving a tree. This has a first step, then a second, ...

BERNOULLI PROCESS: This is a stochastic process in which every step has the same possibilities, and these possibilities have the same probabilities of occurrence on every step. Example: Toss an unfair coin 3 times.

ERASE



1



Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

ERASE

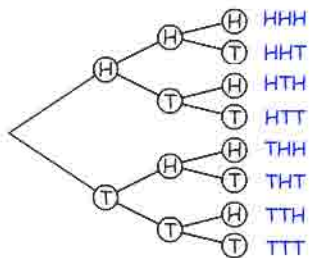


2



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$.



ERASE

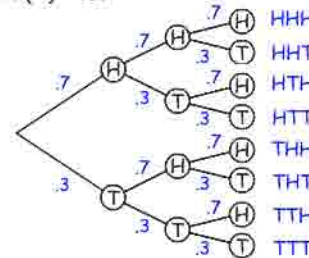


3



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



ERASE

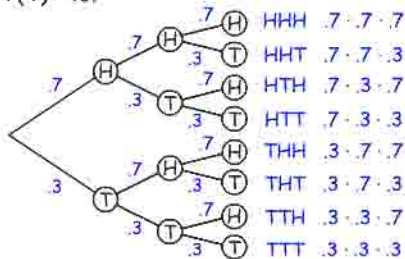


4



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ERASE

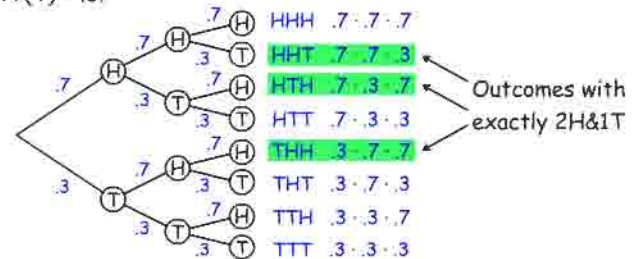


5



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ERASE

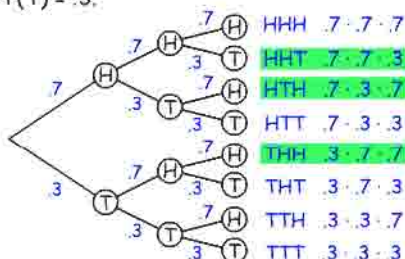


6



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



ANSWER:

$$\begin{aligned} &.7 \cdot .7 \cdot .3 \\ &.7 \cdot .3 \cdot .7 \\ &+.3 \cdot .7 \cdot .7 \\ &3 \cdot (.3 \cdot .7 \cdot .7) \\ &= 3(.3)^1(.7)^2 \end{aligned}$$

ERASE

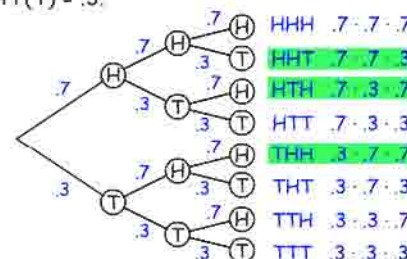


7



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



ANSWER:

$$\begin{aligned} &.7 \cdot .7 \cdot .3 \\ &.7 \cdot .3 \cdot .7 \\ &+.3 \cdot .7 \cdot .7 \\ &3 \cdot (.3 \cdot .7 \cdot .7) \\ &\#tails \quad \#heads \\ &= 3(.3)^1(.7)^2 \\ &\#tosses \quad \Pr(T) \quad \Pr(H) \end{aligned}$$

ERASE



8



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

ERASE



9



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

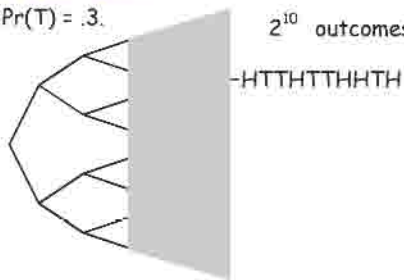
ERASE



10



SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$. 2^{10} outcomes. Here's one:



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

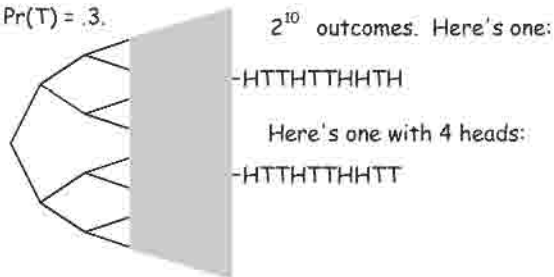
ERASE



11



SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$. 2^{10} outcomes. Here's one:



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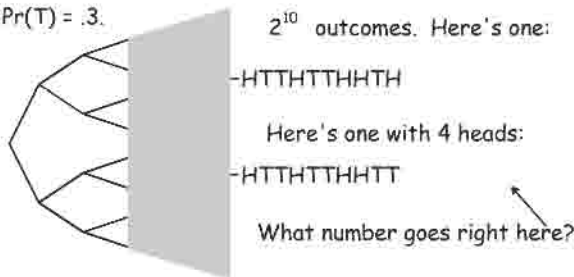
ERASE



12



SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$. 2^{10} outcomes. Here's one:



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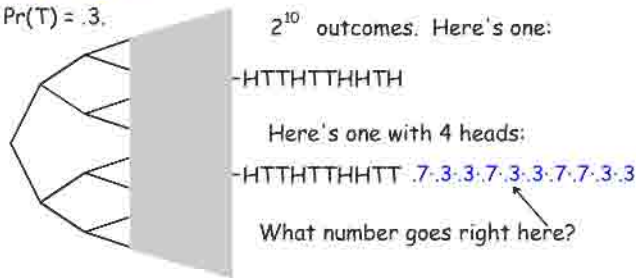
ERASE



13



SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$. 2^{10} outcomes. Here's one:



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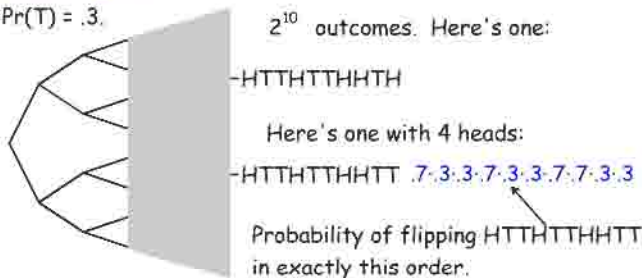
ERASE



14



SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$. 2^{10} outcomes. Here's one:



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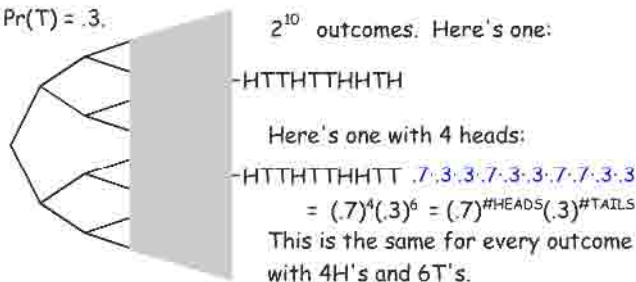
ERASE



15



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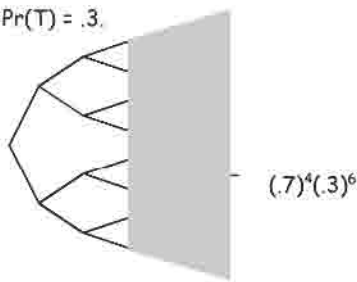
ERASE



16

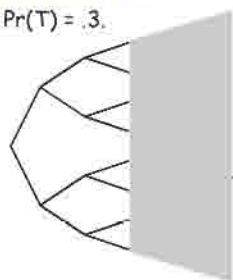


SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



$$(.7)^4(.3)^6$$

How many outcomes have 4H & 6T?

ERASE



17

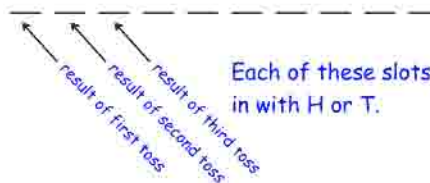


Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?

10 tosses
10 slots



Each of these slots is to be filled in with H or T.

ERASE

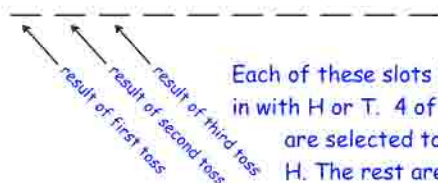


18



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?



Each of these slots is to be filled in with H or T. 4 of the 10 slots are selected to be filled with H. The rest are filled with T.

ERASE

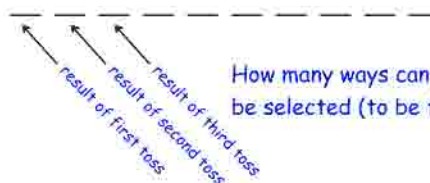


19



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?



How many ways can 4 of the slots be selected (to be filled with H's)?

ERASE

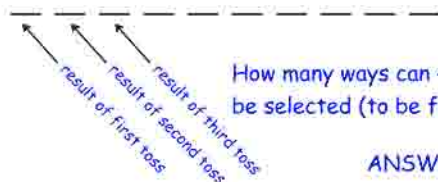


20



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SOLUTION: How many outcomes have 4H & 6T?



How many ways can 4 of the slots be selected (to be filled with H's)?

ANSWER: $C(10,4)$

ERASE



21



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SOLUTION: How many outcomes have 4H & 6T?

Alternate way: How many 10 letter words can be formed using 4 H's and 6 T's?

ANSWER: $C(10,4)$

ERASE



22



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?

Alternate way: How many 10 letter words can be formed using 4 H's and 6 T's?

$$\frac{10!}{6!4!} = C(10,4)$$

ANSWER: $C(10,4)$

ERASE



23



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SOLUTION: Here's a tree. On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



Every outcome with 4H & 6T will have probability $(.7)^4(.3)^6$.

$$(.7)^4(.3)^6$$

$$(.7)^4(.3)^6$$

$$(.7)^4(.3)^6$$

$$(.7)^4(.3)^6$$

$$(.7)^4(.3)^6$$

How many outcomes have 4H & 6T?

ANSWER: $C(10,4)$

ERASE

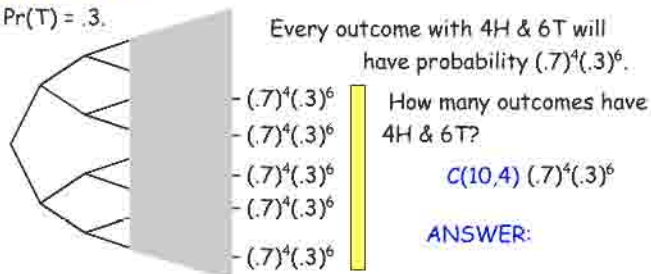


24



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ERASE

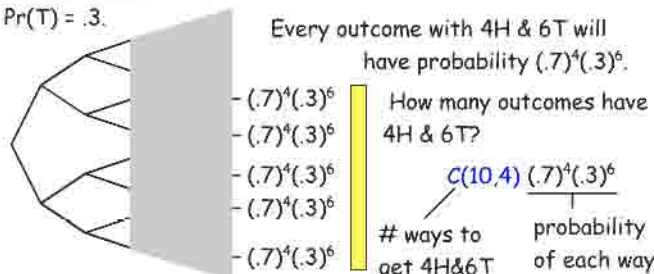


25



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ERASE



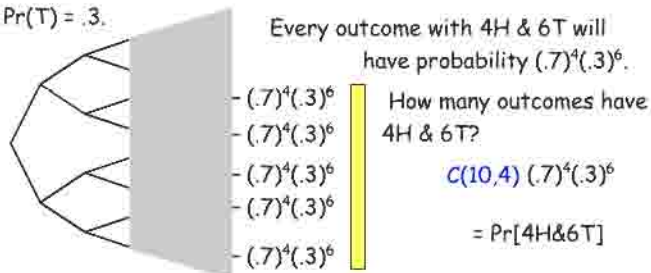
26



Lecture 12 ■ ■

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree On every toss, $\Pr(H) = .7$ & $\Pr(T) = .3$.



ERASE



27



EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 105 times. What is the probability of getting 41 heads (& 64 tails) in the 105 tosses?

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SOLUTION:

$$\Pr(41H \& 64T) =$$

$$C(105,41)(.7)^{41}(.3)^{64}$$

ways to get 41H & 64T probability of each way to get 41H & 64T

ERASE



29



EXAMPLE: Clyde takes a multiple choice test with 20 questions. Each question has 4 choices. He randomly selects one of the 4 choices for every question (i.e. he guesses at random for all answers). What is the probability that he got 11 questions right?

SOLUTION: Each question has one right answer. His probability of success on each question is 1/4. His probability of failure on each question is 3/4. The question asks what is the probability that he got 11S & 9F (11 successes and 9 failures in answering the individual questions).

ERASE



30



EXAMPLE: Clyde takes a multiple choice test with 20 questions. Each question has 4 choices. He randomly selects one of the 4 choices for every question (i.e. he guesses at random for all answers). What is the probability that he got 11 questions right?

SOLUTION: 11S & 9F $\Pr[S] = 1/4$ $\Pr[F] = 3/4$

$$\Pr[11S \& 9F] = C(20,11)(1/4)^{11}(3/4)^9$$

ERASE



31



EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

ERASE



32



EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: $\Pr[S] = 1/4$ $\Pr[F] = 3/4$

$$\Pr[2 \text{ or more sales}] = \Pr[2,3,4 \text{ or } 5 \text{ sales}] \quad \text{disjoint events}$$

$$= \Pr[2 \text{ sales}] + \Pr[3 \text{ sales}] + \Pr[4 \text{ sales}] + \Pr[5 \text{ sales}]$$

ERASE



33



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$$= \Pr[2 \text{ sales}] + \Pr[3 \text{ sales}] + \Pr[4 \text{ sales}] + \Pr[5 \text{ sales}]$$

$$= 1 - \Pr[\{0 \text{ or } 1 \text{ sales}\}] = 1 - \Pr[\{0 \text{ sales}\}] - \Pr[\{1 \text{ sales}\}]$$

ERASE



34



Lecture 12 ■ ■

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: $\Pr[S] = 1/4$ $\Pr[F] = 3/4$

$\Pr[2 \text{ or more sales}] =$

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ERASE



35



EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: $\Pr[S] = 1/4$ $\Pr[F] = 3/4$

$\Pr[2 \text{ or more sales}] = 1 - \Pr[\{0 \text{ sales}\}] - \Pr[\{1 \text{ sales}\}]$

ERASE



36



EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: $\Pr[S] = 1/4$ $\Pr[F] = 3/4$

$\Pr[2 \text{ or more sales}] = 1 - \Pr[\{0 \text{ sales}\}] - \Pr[\{1 \text{ sales}\}]$

$$= 1 - C(5,5) \cdot 3/4 \cdot 3/4 \cdot 3/4 \cdot 3/4 \cdot 3/4 \quad C(5,5) = 1$$

$$- C(5,4) \cdot 3/4 \cdot 3/4 \cdot 3/4 \cdot 3/4 \cdot 1/4 \quad C(5,4) = 5$$

$$= 1 - \frac{243}{1024} - \frac{405}{1024} = \frac{376}{1024}$$

ERASE



37



UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

ERASE



38



UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

$$\Pr[\text{at least 2 heads}] > .8 \Leftrightarrow \Pr[0 \text{ or } 1 \text{ heads}] < .2$$

since $\Pr[0 \text{ or } 1 \text{ heads}] + \Pr[\text{at least 2 heads}] = 1$.

ERASE



39



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since $\Pr[0 \text{ or } 1 \text{ heads}] + \Pr[\text{at least 2 heads}] = 1$.

$$\Pr[0 \text{ or } 1 \text{ heads}] = \Pr[0 \text{ heads}] + \Pr[1 \text{ heads}] =$$

$$= (.7)^n + C(n,1) \cdot (.7)^{n-1} \cdot (.3)$$

$$= (.7)^n + n \cdot (.7)^{n-1} \cdot (.3) < .2 ?$$

ERASE



40



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$$= (.7)^n + n(.7)^{n-1}(.3) < .2 ?$$

ERASE



41



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UGLY SOLUTION: When making n tosses

$$\Pr[\text{at least 2 heads}] > .8 \Leftrightarrow \Pr[0 \text{ or } 1 \text{ heads}] < .2$$

KEEP TRYING
HIGHER VALUES FOR n

$$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$$
$$(.7)^n + n(.7)^{n-1}(.3) < .2 ?$$

ERASE



42



Lecture 12 ■ ■

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UGLY SOLUTION: When making n tosses

$$\Pr[\text{at least 2 heads}] > .8 \Leftrightarrow \Pr[0 \text{ or } 1 \text{ heads}] < .2$$

$$(.7)^8 + 8(.7)^{8-1}(.3) = .2552$$
$$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$$
$$(.7)^n + n(.7)^{n-1}(.3) < .2 ?$$

ERASE



43



UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

$$\Pr[\text{at least 2 heads}] > .8 \Leftrightarrow \Pr[0 \text{ or } 1 \text{ heads}] < .2$$

$$(.7)^9 + 9(.7)^{9-1}(.3) = .1960 < .2$$
$$(.7)^8 + 8(.7)^{8-1}(.3) = .2552$$
$$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$$
$$(.7)^n + n(.7)^{n-1}(.3) < .2 ?$$

ERASE



44



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UGLY SOLUTION: When making n tosses

$$\Pr[\text{at least 2 heads}] > .8 \Leftrightarrow \Pr[0 \text{ or } 1 \text{ heads}] < .2$$

UGLY ANSWER: 9

$$(.7)^9 + 9(.7)^{9-1}(.3) = .1960 < .2$$
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$$(.7)^n + n(.7)^{n-1}(.3) < .2 ?$$

ERASE



45



EXAMPLE: Roll a fair die 3 times. What the probability that a 5 and/or a 6 showed up on exactly 2 tosses?

ERASE



46



EXAMPLE: Roll a fair die 3 times. What is the probability that a 5 and/or a 6 showed up on exactly 2 tosses?

SOLUTION: On each roll $\Pr[5 \text{ or } 6] = 1/3$. On 3 rolls the probability of 2 successes (i.e. two rolls of 5 or 6) is:

$$c(3,2)(1/3)^2(2/3)^1 = 3(1/3)^2(2/3) = 2/9$$

ERASE



47



EXAMPLE: A hat contains 7 red slips of paper and 2 blue slips of paper. A slip is drawn out and its color is noted. Then it is placed back in the hat (i.e. REPLACED). This is repeated 4 more times for a total of 5 draws. What is the probability of drawing a red slip exactly 4 times?

ERASE



48



EXAMPLE: A hat contains 7 red slips of paper and 2 blue slips of paper. A slip is drawn out and its color is noted. Then it is placed back in the hat (i.e. REPLACED). This is repeated 4 more times for a total of 5 draws. What is the probability of drawing a red slip exactly 4 times?

SOLUTION: This is a Bernoulli process where the probability of success (i.e. drawing a red) is $7/9$. What is the probability of 4 successes in 5 trials?

EXAMPLE: A hat contains 7 red slips of paper and 2 blue slips of paper. A slip is drawn out and its color is noted. Then it is placed back in the hat (i.e. REPLACED). This is repeated 4 more times for a total of 5 draws. What is the probability of drawing a red slip exactly 4 times?

SOLUTION: This is a Bernoulli process where the probability of success (i.e. drawing a red) is $7/9$. What is the probability of 4 successes in 5 trials?

$$C(5,4)(7/9)^4(2/9)^1 = 5(7/9)^4(2/9) = 24010/59049$$

Lecture 12 ■ ■

Bernoulli Processes: The probability of obtaining exactly r successes in a Bernoulli process consisting of n trials, each with probability p of success and $q = 1 - p$ of failure is:

$$\Pr[r \text{ successes}] = C(n,r)p^r q^{n-r}$$

for $r = 0, 1, 2, \dots, n$.