BERNOULLI PROCESSES:

STOCHASTIC PROCESS: A stochastic process is a process in which one thing happens after another. Example: Anything involving a tree. This has a first step, then a second, . . .

BERNOULLI PROCESS: This is a stochastic process in which every step has the same possibilities, and these possibilities have the same probabilities of occurence on every step. Example: Toss an unfair coin 3 times.

1

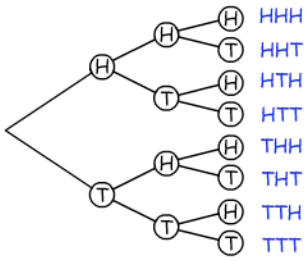
EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

2

Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

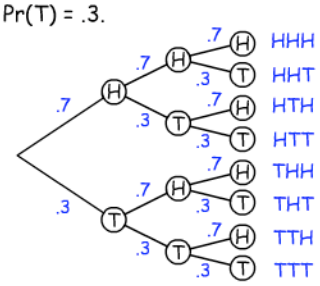
SOLUTION: Here's a tree On every toss, Pr(H) = .7.



3

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



4

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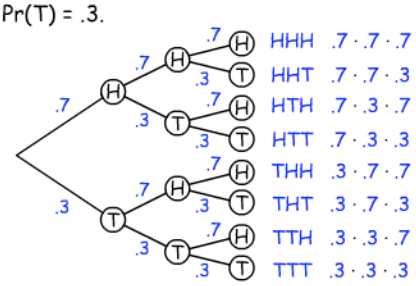

5

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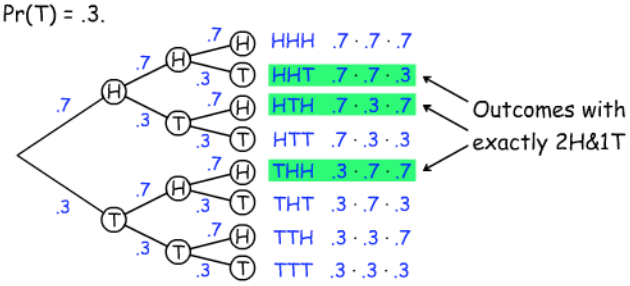
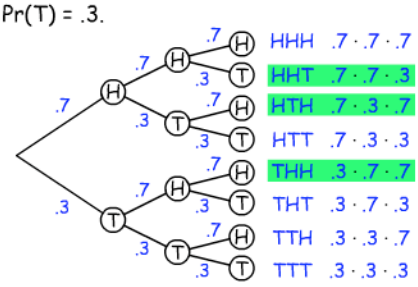


6

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



ANSWER:

$$.7 \cdot .7 \cdot .3$$
$$.7 \cdot .3 \cdot .7$$
$$+ .3 \cdot .7 \cdot .7$$
$$3 \cdot (.3 \cdot .7 \cdot .7)$$

$$= 3(.3)^1(.7)^2$$

7

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 3 times. What is the probability of getting 2 heads (and one tail) in the 3 tosses?

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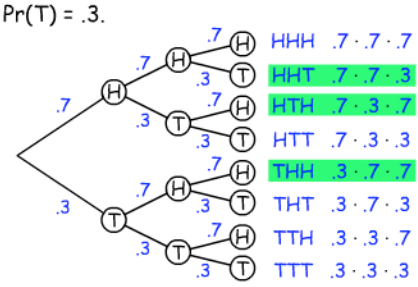


ANSWER:

$$.7 \cdot .7 \cdot .3$$
$$.7 \cdot .3 \cdot .7$$
$$+ .3 \cdot .7 \cdot .7$$
$$3 \cdot (.3 \cdot .7 \cdot .7)$$

$$= 3(.3)^1(.7)^2$$

#tails #heads

#tosses Pr(T) Pr(H)

8

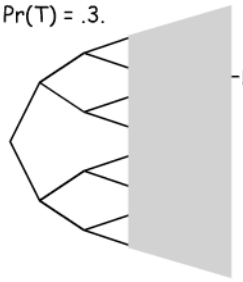
EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

ERASE

9

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



2^{10} outcomes. Here's one:

-HTTHTTHHTH

ERASE

10

Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

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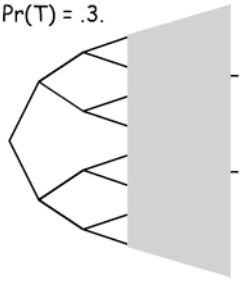


2^{10} outcomes. Here's one:

-HTTHTTHHTH

Here's one with 4 heads:

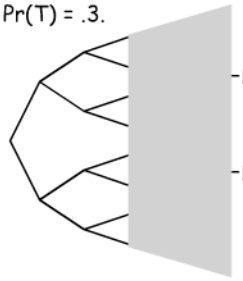
-HTTHTTHHTT

ERASE

11

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



2^{10} outcomes. Here's one:

-HTTHTTHHTH

Here's one with 4 heads:

-HTTHTTHHTT

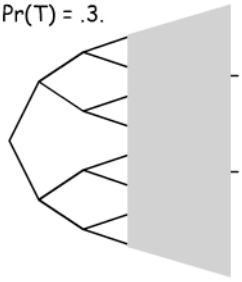
What number goes right here?

ERASE

12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



2^{10} outcomes. Here's one:

-HTTHTTHHTH

Here's one with 4 heads:

-HTTHTTHHTT .7·.3·.3·.7·.3·.3·.7·.7·.3·.3

What number goes right here?

ERASE

13

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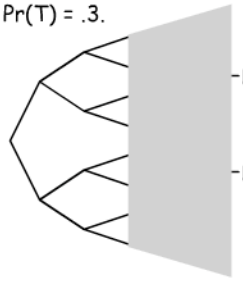

2^{10} outcomes. Here's one:

-HTTHTTHHTH

Here's one with 4 heads:

-HTTHTTHHTT .7·.3·.3·.7·.3·.3·.7·.7·.3·.3

Probability of flipping HTTHTTHHTT in exactly this order.

ERASE

14

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

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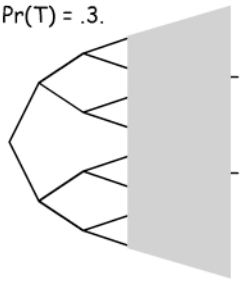

2^{10} outcomes. Here's one:

-HTTHTTHHTH

Here's one with 4 heads:

-HTTHTTHHTT .7·.3·.3·.7·.3·.3·.7·.7·.3·.3

$= (.7)^4(.3)^6 = (.7)^{\#HEADS}(.3)^{\#TAILS}$

This is the same for every outcome with 4H's and 6T's.

ERASE

15

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

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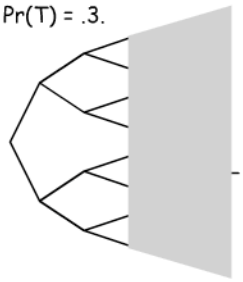

$(.7)^4(.3)^6$

ERASE

16

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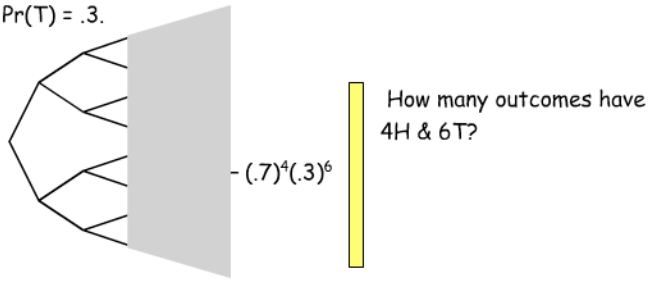
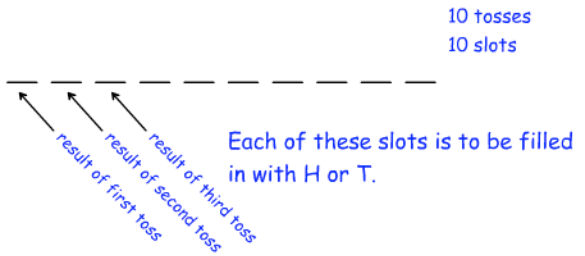


17

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?

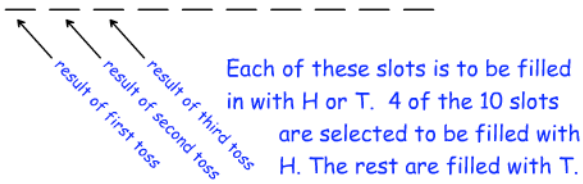


18

Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?



19

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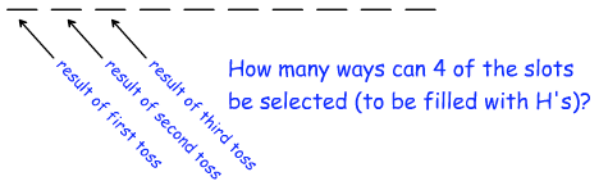

20

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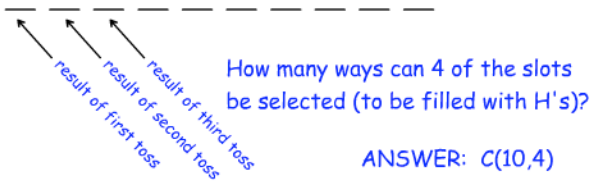

ANSWER: $C(10,4)$

21

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?

Alternate way: How many 10 letter words can be formed using 4 H's and 6 T's?

ANSWER: $C(10,4)$

22

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: How many outcomes have 4H & 6T?

Alternate way: How many 10 letter words can be formed using 4 H's and 6 T's?

$$\frac{10!}{6!4!} = C(10,4)$$

ANSWER: $C(10,4)$

23

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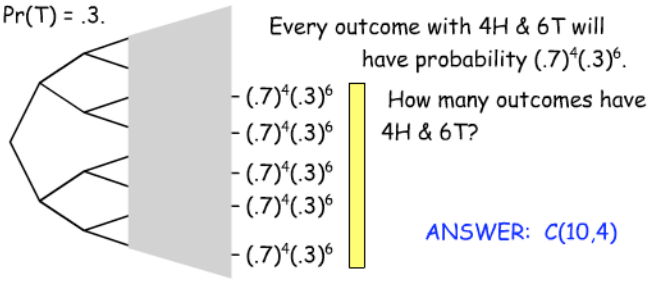

ANSWER: $C(10,4)$

24

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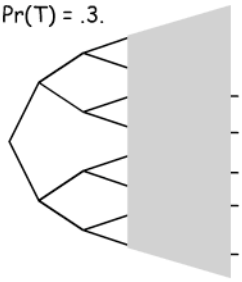

Every outcome with 4H & 6T will have probability $(.7)^4(.3)^6$.

- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$

How many outcomes have 4H & 6T?

$C(10,4)\ (.7)^4(.3)^6$

ANSWER:

25

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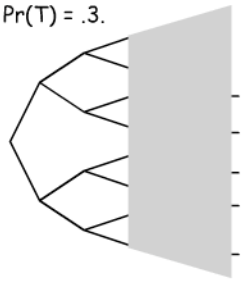


Every outcome with 4H & 6T will have probability $(.7)^4(.3)^6$.

- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$

How many outcomes have 4H & 6T?

$C(10,4)\ (.7)^4(.3)^6$

ways to get 4H&6T

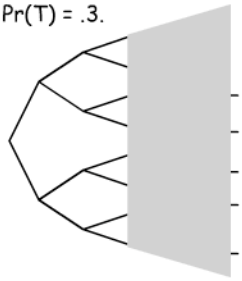
probability of each way

26

Lecture 12

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 10 times. What is the probability of getting 4 heads (and 6 tails) in the 10 tosses?

SOLUTION: Here's a tree On every toss, Pr(H) = .7 & Pr(T) = .3.



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- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$
- $(.7)^4(.3)^6$

How many outcomes have 4H & 6T?

$C(10,4)\ (.7)^4(.3)^6$

= Pr[4H&6T]

27

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 105 times. What is the probability of getting 41 heads (& 64 tails) in the 105 tosses?

28

EXAMPLE: An unfair coin has a probability of .7 of coming up heads on every toss. Toss the coin 105 times. What is the probability of getting 41 heads (& 64 tails) in the 105 tosses?

SOLUTION:

$$\Pr(41H \ \& \ 64T) =$$

$$C(105,41)(.7)^{41}(.3)^{64}$$

ways to get 41H & 64T

probability of each way to get 41H & 64T

29

EXAMPLE: Clyde takes a multiple choice test with 20 questions. Each question has 4 choices. He randomly selects one of the 4 choices for every question (i.e. he guesses at random for all answers). What is the probability that he got 11 questions right?

SOLUTION: Each question has one right answer. His probability of success on each question is 1/4. His probability of failure on each question is 3/4. The question asks what is the probability that he got 11S & 9F (11 successes and 9 failures in answering the individual questions).

30

EXAMPLE: Clyde takes a multiple choice test with 20 questions. Each question has 4 choices. He randomly selects one of the 4 choices for every question (i.e. he guesses at random for all answers). What is the probability that he got 11 questions right?

SOLUTION: 11S & 9F Pr[S] = 1/4 Pr[F] = 3/4

$$\Pr[11S \ \& \ 9F] = C(20,11)(1/4)^{11}(3/4)^9$$

31

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

32

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: Pr[S] = 1/4 Pr[F] = 3/4

Pr[2 or more sales] = Pr[2,3,4 or 5 sales] disjoint events

= Pr[2 sales] +Pr[3 sales] +Pr[4 sales] +Pr[5 sales]

33

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: Pr[S] = 1/4 Pr[F] = 3/4

Pr[2 or more sales] = Pr[2,3,4 or 5 sales] disjoint events

= Pr[2 sales] +Pr[3 sales] +Pr[4 sales] +Pr[5 sales]

= 1 - Pr[(0 or 1 sales)] = 1 - Pr[(0 sales)] - Pr[(1 sales)]

34

Lecture 12

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: Pr[S] = 1/4 Pr[F] = 3/4

Pr[2 or more sales] =

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35

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36

EXAMPLE: A sales representative gets a sale on 1/4 of his business calls. What is the probability that at least 2 sales will result from 5 calls?

SOLUTION: Pr[S] = 1/4 Pr[F] = 3/4

Pr[2 or more sales] = 1 - Pr[(0 sales)] - Pr[(1 sales)]

$= 1 - C(5,5)\cdot 3/4\cdot 3/4\cdot 3/4\cdot 3/4\cdot 3/4 - C(5,4)\cdot 3/4\cdot 3/4\cdot 3/4\cdot 3/4\cdot 1/4$

$C(5,5) = 1$
$C(5,4) = 5$

$= 1 - \frac{243}{1024} - \frac{405}{1024} = \frac{376}{1024}$

37

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

38

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 $\Leftrightarrow$ Pr[0 or 1 heads] < .2

since Pr[0 or 1 heads] + Pr[at least 2 heads] = 1.

39

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 $\Leftrightarrow$ Pr[0 or 1 heads] < .2

since Pr[0 or 1 heads] + Pr[at least 2 heads] = 1.

Pr[0 or 1 heads] = Pr[0 heads] + Pr[1 heads] =

$= (.7)^n + C(n,1)(.7)^{n-1}(.3)$

$= (.7)^n + n(.7)^{n-1}(.3) < .2$?

40

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$= (.7)^n + n(.7)^{n-1}(.3) < .2$?

41

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UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 ⇔ Pr[0 or 1 heads] < .2

KEEP TRYING
HIGHER VALUES FOR n

$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$
$(.7)^n + n(.7)^{n-1}(.3) < .2$?

42

Lecture 12

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 ⇔ Pr[0 or 1 heads] < .2

$(.7)^8 + 8(.7)^{8-1}(.3) = .2552$
$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$
$(.7)^n + n(.7)^{n-1}(.3) < .2$?

43

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 ⇔ Pr[0 or 1 heads] < .2

$(.7)^9 + 9(.7)^{9-1}(.3) = .1960 < .2$
$(.7)^8 + 8(.7)^{8-1}(.3) = .2552$
$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$
$(.7)^n + n(.7)^{n-1}(.3) < .2$?

44

UGLY EXAMPLE: An unfair coin has a probability of .3 landing heads on each toss. What is the least number of times that you could flip the coin to have a probability of more than .8 of tossing at least 2 heads?

UGLY SOLUTION: When making n tosses

Pr[at least 2 heads] > .8 ⇔ Pr[0 or 1 heads] < .2

UGLY ANSWER: 9

$(.7)^9 + 9(.7)^{9-1}(.3) = .1960 < .2$
$(.7)^8 + 8(.7)^{8-1}(.3) = .2552$
$(.7)^4 + 4(.7)^{4-1}(.3) = .6517$
$(.7)^n + n(.7)^{n-1}(.3) < .2$?

45

EXAMPLE: Roll a fair die 3 times. What the probability that a 5 and/or a 6 showed up on exactly 2 tosses?

46

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SOLUTION: On each roll Pr[5 or 6] = 1/3. On 3 rolls the probability of 2 successes (i.e. two rolls of 5 or 6) is:

$C(3,2)(1/3)^2(2/3)^1 = 3(1/3)^2(2/3) = 2/9$

47

EXAMPLE: A hat contains 7 red slips of paper and 2 blue slips of paper. A slip is drawn out and its color is noted. Then it is placed back in the hat (i.e. REPLACED). This is repeated 4 more times for a total of 5 draws. What is the probability of drawing a red slip exactly 4 times?

48

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SOLUTION: This is a Bernoulli process where the probability of success (i.e. drawing a red) is 7/9. What is the probability of 4 successes in 5 trials?

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SOLUTION: This is a Bernoulli process where the probability of success (i.e. drawing a red) is 7/9. What is the probability of 4 successes in 5 trials?

$$C(5,4)(7/9)^4(2/9)^1 = 5(7/9)^4(2/9) = 24010/59049$$

Bernoulli Processes: The probability of obtaining exactly r successes in a Bernoulli process consisting of n trials, each with probability p of success and q = 1 - p of failure is:

$$\Pr[r \text{ successes}] = C(n,r)p^r q^{n-r}$$

for r = 0, 1, 2, . . . , n.