1) Which of the following is a correct statement about the three lines given by the equations:

$$① x + y = 2, ② y = 5, ③ x + 2y = 7.$$

(a) two of these lines are parallel.
(b) two of these lines have a positive slope, and the slope is undefined for the other line.
(c) none of these lines are parallel, but they do not all go through the same point.
(d) These lines all go through a single point
(e) none of the above.

$$\left.\begin{array}{lll} \text{SLOPES: } ① & y = -x + 2 & \text{slope} = -1 \\ ② & y = 5 & \text{slope} = 0 \\ ③ & y = -\tfrac{1}{2}x + \tfrac{7}{2} & \text{slope} = -\tfrac{1}{2} \end{array}\right\} \begin{array}{l} \text{NO TWO SLOPES} \\ \text{ARE EQUAL} \\ \Rightarrow \\ \text{NO TWO LINES} \\ \text{ARE PARALLEL} \end{array}$$

a) doesn't hold
b) doesn't hold (two of the lines have negative slope)
c) + d) One of these will be true.
Try to find a point of intersection for all 3 lines:

Substitute ② into ① + ③:

$$\left.\begin{array}{lll} ① & x + 5 = 2 \Rightarrow & x = -3 \\ ③ & x + 2 \cdot 5 = 7 \Rightarrow & x = -3 \end{array}\right\} (-3, 5) \text{ satisfies } ①, ②, ③.$$

d) is true.

For each of the augmented matrices in the next three problems, determine which of the following statements is true about the associated system of linear equations:

(a) The system has no solution.
(b) The system has exactly one solution.
(c) The system has infinitely many solutions in which one variable can be selected arbitrarily.
(d) The system has infinitely many solutions in which two variables can be selected arbitrarily.
(e) none of the above.

2) $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ -1 & -1 & -4 \\ 2 & 2 & 8 \\ 1 & 3 & 6 \end{array}\right] \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 2 & 2 & 8 \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{-2R_1+R_3} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \\ 1 & 3 & 6 \end{pmatrix} \xrightarrow{-R_1+R_4} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

$\xrightarrow{2R_2+R_3} \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{-R_1+R_4} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$

Each variable x_1, x_2 has a pivot in its column $\Rightarrow$ no free variables $\Rightarrow$ b)

3) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 1 & 2 & 0 & 12 \\ -1 & -1 & 1 & -7 \end{array}\right]$

4) $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & -3 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R_1+R_3} \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$

$\xrightarrow{-R_2+R_3} \begin{array}{c} \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$

x_1 and x_3 are pivot variables
x_2 and x_4 are free.
Answer: d)

$\begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 1 & 2 & 0 & 12 \\ -1 & -1 & 1 & -7 \end{pmatrix} \xrightarrow{R_1+R_4} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 1 & 2 & 0 & 12 \\ 0 & -1 & -1 & -4 \end{pmatrix} \xrightarrow{-R_1+R_3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 6 \\ 0 & -1 & -1 & -4 \end{pmatrix}$

$\xrightarrow{-2R_2+R_3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & -1 & -1 & -4 \end{pmatrix} \longrightarrow 0x_1 + 0x_2 + 0x_3 = -2$

$\Rightarrow$ NO SOLUTION a)

5) A certain 3×3 matrix A has as its inverse the matrix

$$A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}.$$

If

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix},$$

then what is the value of x?

(a) 1
(b) 15
(c) -1
(d) 20
(e) none of the above

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \Rightarrow A^{-1} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$$

$$A^{-1} \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 4 + (-3)(-5) + 1 \cdot 3 \\ * \\ * \end{pmatrix} = \begin{pmatrix} 20 \\ * \\ * \end{pmatrix}$$

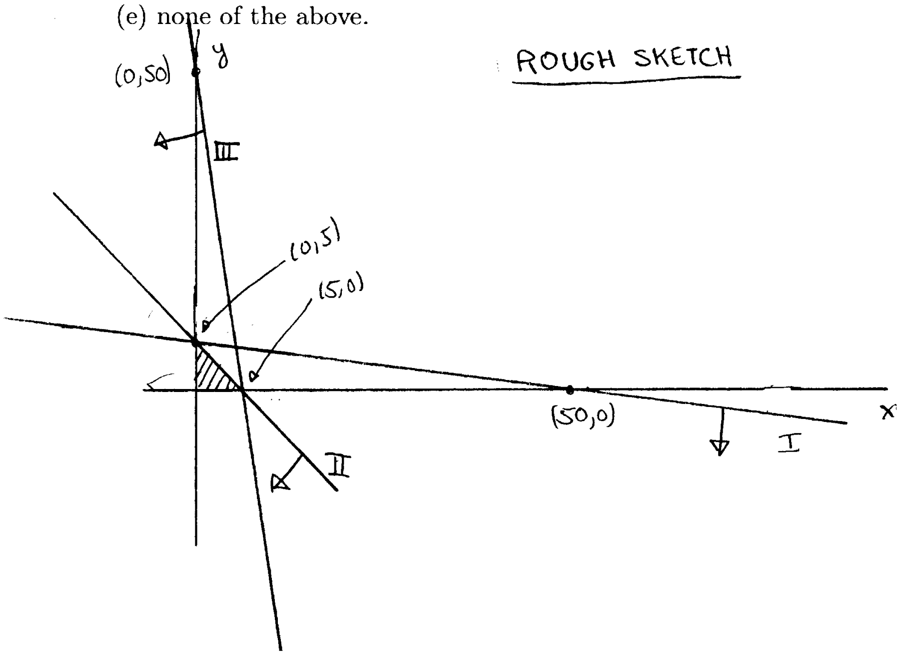
$$\underline{\underline{x = 20}}$$

6) Find the maximum value of $2x + y$ on the feasible set given by the constraints:

$$\underset{\text{I}}{x + 10y \le 50}, \quad \underset{\text{II}}{x + y \le 5}, \quad \underset{\text{III}}{10x + y \le 50}, \quad x \ge 0, \quad y \ge 0.$$

(a) 1
(b) 10
(c) 20
(d) 30
(e) none of the above.

ROUGH SKETCH



Corner points	$2x+y$
$(0,5)$	5
$(5,0)$	10 ← Maximum
$(0,0)$	0

Answer: 10

7) Determine the matrix product

$$\begin{bmatrix} -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 1 & 3 \end{bmatrix}$$

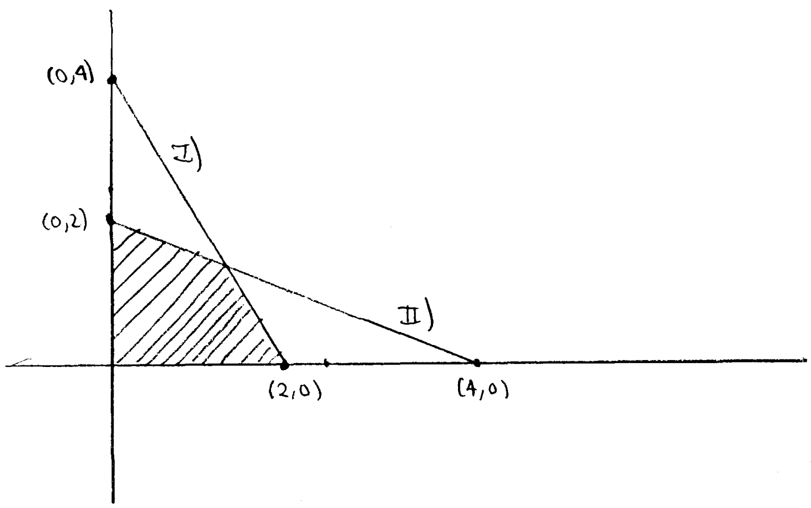
(a) $\begin{bmatrix} 3 & 0 \end{bmatrix}$
(b) $\begin{bmatrix} -3 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 3 \end{bmatrix}$
(d) $\begin{bmatrix} 1 & 4 \end{bmatrix}$
(e) none of the above.

Answer: c)

8) Which of the following is not a corner point of the feasible set given by $x \geq 0$, $y \geq 0$, $\underbrace{2x + y \leq 4}_{\text{I)}}$,
II) $x + 2y \leq 4$, $2x + 2y \leq 5$. III)

DRAW ONLY IN 1st QUADRANT

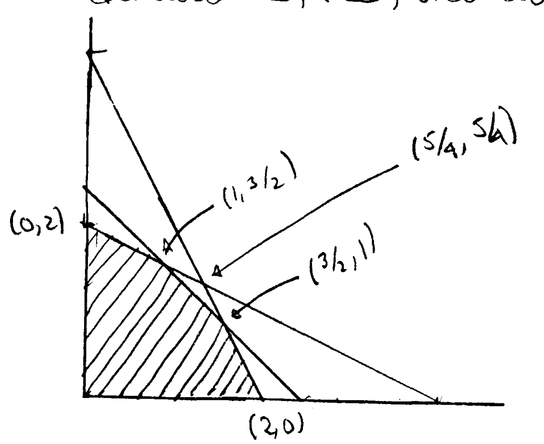
(a) $(2, 0)$
(b) $(0, 2)$
(c) $(1, 3/2)$
(d) $(4/3, 4/3)$
(e) none of the above.



The feasible set will be the shaded area possibly restricted further by III). II) + III) intersect:

$$\begin{array}{r} 2x + 2y = 5 \\ x + 2y = 4 \\ \hline x \quad = 1 \end{array} \Rightarrow y = 3/2$$

Likewise I) + III) intersect at $(3/2, 1)$. The picture looks like:



The point $(4/3, 4/3)$ isn't even a "point of intersection." Also it doesn't even satisfy III.

Answer: $(4/3, 4/3)$

9) Given a system of 3 linear equations in 3 variables, which of the following can never occur?
(a) There are infinitely many solutions.
(b) There is exactly one solution.
(c) There are exactly three solutions.
(d) There are no solutions.
(e) None of the above.

c)

If there is more than one solution, there has to be at least one free variable. $\Rightarrow$ ∞ no. of solutions (i.e. <u>not</u> 3).

10) A cook in Chet's Diner uses mixed vegetables to prepare stir fried vegetables as side dishes and to make vegetable soup. It takes 32 ounces of vegetables and 10 minutes of time to produce 10 servings of stir fried vegetables, and it takes 48 ounces of vegetables and 20 minutes of time to make a pot of soup which produces 8 bowls. The profit on one order of stir fried vegetables is \$.30 and the profit on one bowl of soup is \$.50. There are 160 ounces of vegetables and 60 minutes of time available. We wish to figure out how much of each food should be produced to yield maximum profit.

Which of the following is the objective funtion when this situation is formulated as a linear programming problem:

Let x be the number of orders of stir fried vegetables and y be the number of bowls of soup.

(a) $10x + 20y$
(b) $32x + 48y$
(c) $.30x + .50y$
(d) $.50x + .30y$
(e) none of the above.

\# STIR FRY ORDERS

$$.3x + .5y$$

\# BOWLS OF SOUP

PROFIT/STIR FRY ORDER

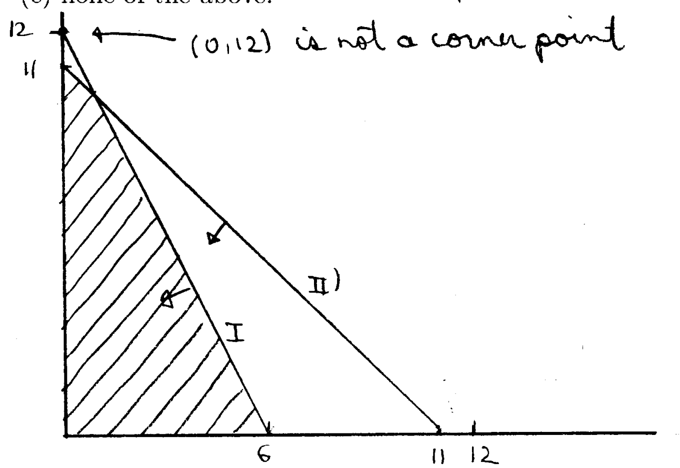
PROFIT/BOWL SOUP

11) A feasible set for a linear programming problem is defined by

$$\left.\begin{array}{r} x \geq 0 \\ y \geq 0 \end{array}\right\} \text{1ST QUADRANT}$$

$$\begin{array}{rl} 2x + y \leq 12 & \text{I)} \\ 4x + 4y \leq 44 & \text{II)} \end{array}$$

Which of the following is not a corner point of the feasible set?
(a) $(1, 10)$
(b) $(0, 12)$
(c) $(0, 0)$
(d) $(0, 11)$
(e) none of the above.



12) A Markov chain has the transition matrix

$$P = \begin{pmatrix} 0 & 2/5 & 3/5 \\ 1/5 & 4/5 & 0 \\ 2/5 & 1/5 & 2/5 \end{pmatrix}$$

and an initial state vector

$$X_0 = [1/6\ 1/3\ 1/2]\ .$$

What is the probability that this chain will be in state 1 after two transitions?

$$\begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \qquad\qquad\qquad\qquad \begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\begin{pmatrix} 1/6 & 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} & P^2 & \end{pmatrix} = \begin{pmatrix} \cdot\,, & \cdot\,, & \cdot \end{pmatrix}$$

FIND THIS

2 TRANSITIONS

Only the first column of P^2 is needed to make this calculation

$$\underset{P}{\begin{pmatrix} 0 & 2/5 & 3/5 \\ 1/5 & 4/5 & 0 \\ 2/5 & 1/5 & 2/5 \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 2/5 & 3/5 \\ 1/5 & 4/5 & 0 \\ 2/5 & 1/5 & 2/5 \end{pmatrix}} = \underset{P^2}{\begin{pmatrix} 8/25 & * & * \\ 4/25 & * & * \\ 5/25 & * & * \end{pmatrix}}$$

$$\begin{pmatrix} 1/6 & 1/3 & 1/2 \end{pmatrix} \begin{pmatrix} 8/25 \\ 4/25 \\ 5/25 \end{pmatrix} = \frac{8}{150} + \frac{8}{150} + \frac{15}{150} = \frac{31}{150}$$ ← Answer

13) A Markov chain has the transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}.$$

What is the first entry of the stable vector for this transition matrix?

$$wP = w \Rightarrow P^T w^T = w^T \Rightarrow (P^T - I) w^T = 0.$$

$$P^T = \begin{pmatrix} 1/2 & 1/3 & 0 \\ 0 & 2/3 & 2/3 \\ 1/2 & 0 & 1/3 \end{pmatrix} \Rightarrow P^T - I = \begin{pmatrix} -1/2 & 1/3 & 0 \\ 0 & -1/3 & 2/3 \\ 1/2 & 0 & -2/3 \end{pmatrix}$$

$$\begin{pmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & 0 \\ 1/2 & 0 & -2/3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[2R_1+R_4]{R_1+R_3} \begin{pmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & 0 \\ 0 & 1/3 & -2/3 & 0 \\ 0 & 5/3 & 1 & 1 \end{pmatrix} \xrightarrow{-5R_3+R_4}$$

$$\begin{pmatrix} -1/2 & 1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & 0 \\ 0 & 1/3 & -2/3 & 0 \\ 0 & 0 & 13/3 & 1 \end{pmatrix} \begin{matrix} \longrightarrow -1/3\, w_2 + 2/3\, w_3 = 0 \Rightarrow -1/3\, w_2 + 2/3 \cdot 3/13 = 0 \\ \Rightarrow w_2 = 6/13 \\ \longrightarrow 13/3\, w_3 = 1 \Rightarrow w_3 = 3/13 \end{matrix}$$

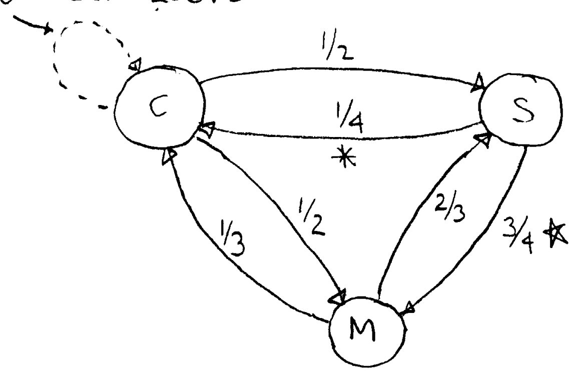
$$1 = w_1 + w_2 + w_3 = w_1 + 6/13 + 3/13 \Rightarrow w_1 = 4/13.$$

Check:

$$(4/13, 6/13, 3/13) \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (4/13, 6/13, 3/13)$$

14) A purchasing manager for a construction company purchases gravel from three companies Cox Gravel, Miller Rock, Smith Stone. In order to maintain competition between them and keep prices low, for each purchase she selects one of the suppliers at random according to the following rules: She never orders from the same supplier two times in succession. If one purchase is from Cox, then for the next purchase she is equally likely to select Miller or Smith. If one purchase is from Miller, then for the next purchase she is two times more likely to select Smith than to select Cox. If one purchase is from Smith, then for the next purchase she is three times more likely to select Miller than to select Cox. View this process as a Markov chain with state 1 being a purchase from Cox, state 2 being a purchase from Miller, and state 3 being a purchase from Smith. What is the transition matrix for this Markov chain?

NO "SELF" LOOPS



Transition diagram.

Notice * is 3 times * as required by the underlined statement

Transition matrix

$$\begin{array}{c} \\ C \\ S \\ M \end{array} \begin{array}{c} \begin{array}{ccc} C & S & M \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 1/3 & 2/3 & 0 \end{pmatrix} \end{array}$$

order is arbitrary as long as it is consistent from row to column.

15) A Markov chain has the transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Is it regular?

$$\underset{P}{\begin{pmatrix} 0 & 0 & \ast \\ \ast & 0 & \ast \\ 0 & \ast & 0 \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 0 & \ast \\ \ast & 0 & \ast \\ 0 & \ast & 0 \end{pmatrix}} = \underset{P^2}{\begin{pmatrix} 0 & \ast & 0 \\ 0 & \ast & \ast \\ \ast & 0 & \ast \end{pmatrix}}$$

$$\underset{P^2}{\begin{pmatrix} 0 & \ast & 0 \\ 0 & \ast & \ast \\ \ast & 0 & \ast \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 0 & \ast \\ \ast & 0 & \ast \\ 0 & \ast & 0 \end{pmatrix}} = \underset{P^3}{\begin{pmatrix} \ast & 0 & \ast \\ \ast & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix}}$$

$$\underset{P^3}{\begin{pmatrix} \ast & 0 & \ast \\ \ast & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 0 & \ast \\ \ast & 0 & \ast \\ 0 & \ast & 0 \end{pmatrix}} = \underset{P^4}{\begin{pmatrix} 0 & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix}}$$

$$\underset{P^4}{\begin{pmatrix} 0 & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix}} \underset{P}{\begin{pmatrix} 0 & 0 & \ast \\ \ast & 0 & \ast \\ 0 & \ast & 0 \end{pmatrix}} = \underset{P^5}{\begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix}}$$

Yes, it is regular. P^5 has no zero entries.

16) The president of the bank must select one person to manage the East Gate branch, another person to manage the North Mall branch, and a third person to manage the West Shore branch. She will choose these managers from a group of 9 assistant managers. How many different outcomes axe possible?

(A) 19,683	(B) 524	(C) 504	(D) 729	E) 27
(F) 84	(G) 24	(H) 1	(J) none of the others	

EASTGATE　　NORTH MALL　　WEST SHORE

(___ , ___ , ___)

9 choices　　8 choices　　7 choices

$9\cdot 8\cdot 7 = \underline{\underline{504}}$　　or　　$P(9,3) = 9\cdot 8\cdot 7.$

17) An interior decorator must select a color of paint for each of apartments A, B, and C. If he chooses from 5 shades of off-white, 4 shades of yellow, and 3 shades of blue, how many different outcomes are possible?

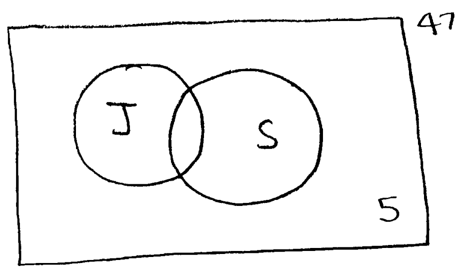
(A) 1,320 (B) 220 (C) 60 (D) 33 (E) 3
(F) 1,728 (G) 531,441 (H) 12 (J) 36 (K) 216
(L) none of the others

$5+4+3 = 12$ choices for each apartment

$$\underbrace{12}_{\#\text{ ways to choose a color for A}} \cdot \underbrace{12}_{B} \cdot \underbrace{12}_{C} = \underline{\underline{1728}}$$

18) When the 47 partners of the law firm met to determine whether to promote John or Susan (or both) from associate to partner, 34 favored promoting John, 31 favored promoting Susan, and 5 opposed both promotions. How many favored promoting both John and Susan?

(A) 11 (B) 18 (C) 13 (D) 23 (E) 8
(F) 28 (G) 5 (H) 33 (i) 19 (K) 42
(L) 16 (M) none of the others



$n(J \cup S) + 5 = 47$

$\Rightarrow n(J \cup S) = 42.$

But $42 = n(J \cup S) = \underbrace{n(J)}_{34} + \underbrace{n(S)}_{31} - n(J \cap S)$

$\Rightarrow 42 = 34 + 31 - n(J \cap S) \Rightarrow n(J \cap S) = \underline{\underline{23}}.$

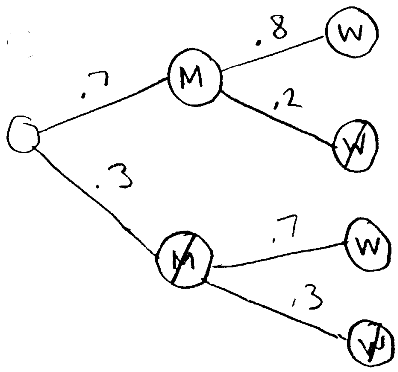
M

19) 70 % of the Chamber's members favor mandatory health insurance and the rest oppose it. 80 % of those who favor mandatory health insurance favor welfare reform. 70 % of those who oppose mandatory health insurance favor welfare reform. If a member of the chamber is selected at random, what ~~IES~~ is the probability that this person does not favor welfare reform?

(A) 0.75	(B) 0.35	(C) 0.77	(D) 0.65	(E) 0.25
(F) 0.16	(G) 0.35	(H) 0.23	(J) none of the others	

This is a conditional probability. It is a tip-off to use a tree. Also, the 80% (or .80) will appear on the 2nd or higher branch of the tree).

W= welfare reform



$.7 \cdot .2 = .14$

Do not favor welfare reform

$.3 \cdot .3 = .09$

$.23$ ← Answer

20) A museum's curator is to select 3 still lifes and 2 portraits by a certain artist for an anniversary show. The museum has 7 still lifes and 5 portraits by this artist. How many different outcomes are possible?

(A) 95,040	(B) 5	(C) 1	(D) 8,575	(E) 4,200
(F) 31	(G) 350	(H) 45	(J) 230	(K) .368
(L) 210	(M) 792	(N) 60	(P) none of the others	

$$\binom{7}{3}\binom{5}{2} = \frac{7!}{4!\,3!}\,\frac{5!}{3!\,2!} = \frac{7\cdot 6\cdot 5}{3!}\,\frac{5\cdot 4}{2!} = 350$$

from the 7 still lifes, choose 3

From the 5 portraits, choose 2.

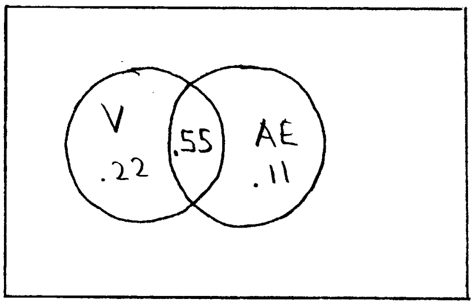
IF the problem were to select 3 still lifes OR 2 portraits

then the answer would be

$$\binom{7}{3} + \binom{5}{2} = 35 + 10 = 45.$$

But this is not the problem.

21) 77% of a group of travelers have Visa cards, 66% have American Express cards, and 55% have both. If one of these travelers, selected at random, does not have a Visa card, what is the probability that this traveler also does not have an American Express card?



Restatement of question

$Pr((AE)^c \mid V^c) = ?$

$$Pr((AE)^c \mid V^c) = \frac{Pr((AE)^c \cap V^c)}{Pr(V^c)}$$

This is easy $1 - .77 = .23$

$(AE)^c \cap V^c = ((AE) \cup V)^c$ deMorgan's law

$Pr((AE) \cup V) = .22 + .55 + .11 = .88$

$\Rightarrow Pr(((AE) \cup V)^c) = 1 - .88 = .12 \quad (= Pr((AE)^c \cap V^c))$

answer:

$$\frac{.12}{.23} = \frac{12}{32}$$

22) In a certain computer game, the player's first oponent is either a dragon or a monster. Each time the game is played, the computer randomly and independently selects whether the first opponent is a dragon or a monster, and the probability that it selects a dragon is 4/7. If 4 people play this game, what is the probability that the first opponent is a dragon for three of them?

This is a Bernoulli process.

$$\binom{4}{3}\left(4/7\right)^3\left(3/7\right)^1$$

23) A bin contains 7 rolls of Kodak film and 5 rolls of Fuji film. If a customer selects 4 rolls simutaneously from this bin, how many different possible outcomes have at least one roll of Fuji film and at least one roll of Kodak film? (Each roll is a distinct object.)

$$\binom{12}{4} = \frac{12!}{8!4!} = \frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1} = 11\cdot 5\cdot 9 = 495$$

= # ways to choose 4 of the 12 film rolls. However this includes getting 4 Fuji or 4 Kodak.

$\binom{7}{4}$ = # ways to choose 4 Kodak $= \frac{7!}{3!4!} = \frac{7\cdot 6\cdot 5}{3!} = 35$

$\binom{5}{4}$ = # ways to choose 4 Fuji $= \frac{5!}{1!4!} = 5$.

Answer:

$$495 - 35 - 5 = 455$$

24) How many distinct numbers ar there with digits $\underbrace{9,8,8,6,7,7,4,4}_{\text{8 numbers.}}$ that are even (e.g. 98867744 is one, 79884746 is another)?

How many distinct numbers of any kind?

$$\frac{8!}{2!\,2!\,2!}$$

TWO 8's

TWO 7's

TWO 4's

How many of these are odd (i.e end in 9 or 7)?

End in 9: $\overbrace{\underbrace{8867744}_{\text{Permute}}}^{\text{7 numbers}}\ \underline{9}$

$$\boxed{\frac{7!}{2!\,2!\,2!}}$$

TWO 8's 7's 4's

End in 7: $\overbrace{\underbrace{8867449}_{\text{PERMUTE}}}^{\text{7 numbers}}\ \underline{7}$

$$\boxed{\frac{7!}{2!\,2!}}$$

TWO 8's TWO 4's

Answer: $\dfrac{8!}{2!2!2!} - \dfrac{7!}{2!2!2!} - \dfrac{7!}{2!2!} = \dfrac{5\cdot 7!}{2!2!2!}$

$= 3150$

25a) Let $A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$. Find a 2×2 matrix B such that

$$ABA^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

multiply both sides on the left by A^{-1} and on the right by A

$$A^{-1}ABA^{-1}A = A^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}A$$

$$B = A^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}A = \underbrace{\begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 7 & 1 \end{pmatrix}$$

Answer: $B = \begin{pmatrix} -1 & 0 \\ 7 & 1 \end{pmatrix}$

b) Let A be the same matrix as in part a). Find a 2×2 matrix B such that

$$(A+I)B - B = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}.$$

$$(A+I)B - B = AB + IB - B = AB + B - B = AB$$

So $AB = I \Rightarrow B = A^{-1}$

Answer: $B = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$

26) On any given toss, an unfair coin has a probability of .7 of landing heads. The coin is tossed nine (9) times. What is the probability of getting 3 heads and 6 tails (<u>in any order</u>)? You may leave your answer as a product of numbers (e.g. $102(.7)^8(.4)^3$ is an answer in acceptable form).

$$\binom{9}{3}(.7)^3(.3)^6$$

Answer: $Pr[3H\&6T] = 84(.7)^3(.3)^6$

27) On any given toss, an unfair coin that is VERY thick has a probability of .5 of landing heads, a probability of .3 of landing tails, and a probability of .2 of landing on its side. The coin is tossed nine (9) times. What is the probability of getting 6 heads and 2 tails and one "sider" (<u>in any order</u>)? The answer is (almost) given below

3·84

$$Pr[6H\&2T\&1S] = \underline{\ 252\ }\,(.5)^6(.3)^2(.2)^1$$

except that a number has to be placed in the blank. Fill in that blank (with a number).

Choose 1 position for S from the 1 position left over

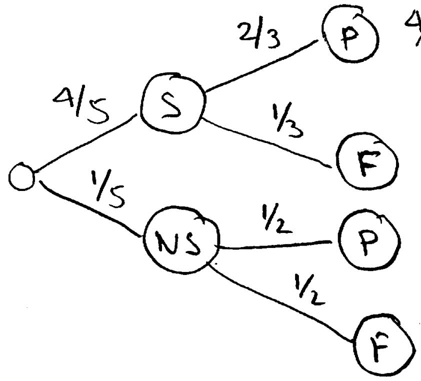
$$\underbrace{\binom{9}{6}}_{=84}\underbrace{\binom{3}{2}}_{=3}\binom{1}{1}(.5)^6(.3)^2(.2)^1$$

Choose 6 of the 9 "positions" for H

Choose 2 of the remaining 3 positions for T

28) Josie studies for 4/5 of her exams. <u>If she studies, she has a 2/3 probability of passing</u>. If she doesn't study, she has a 1/2 probability of passing.

A conditional statement is <u>usually</u> a tip-off to use a tree



a) $\frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$

$\frac{1}{5}\cdot\frac{1}{2} = \frac{1}{10}$

$Pr(P) = \frac{8}{15} + \frac{1}{10}$

$= \frac{19}{30}$ ☆

a) You select, at random, one of the exams that she has taken. What is the probability that she studied for that exam and passed it?

Answer: $\underline{8/15}$

b) You select, at random, one of the exams that she has taken, and you see that she has passed it. What is the probability that she studied for the exam?

FROM PART a)

$$Pr(S \mid P) = \frac{Pr(S\cap P)}{Pr(P)} = \frac{8/15}{19/30}$$

☆

Answer: $\underline{16/19}$

29) In the following, k is some number:

$$\begin{aligned} 2x_1 + 1x_2 + 2x_3 + 2x_4 - 4x_5 &= k \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 4x_5 &= -3 \\ 2x_1 + 1x_2 + 2x_3 + 3x_4 - 4x_5 &= 0\,. \end{aligned}$$

Solve this system for x_2.

(A) $k+3$ (B) $-k-3$ (C) $-k+3$ (D) -3 (E) 0

(F) none of the others

Form the augmented matrix and row reduce.

$$\begin{pmatrix} 2 & 1 & 2 & 2 & -4 & k \\ 2 & 2 & 2 & 3 & -4 & -3 \\ 2 & 1 & 2 & 3 & -4 & 0 \end{pmatrix} \xrightarrow{-R_1+R_2} \begin{pmatrix} 2 & 1 & 2 & 2 & -4 & k \\ 0 & 1 & 0 & 1 & 0 & -k-3 \\ 2 & 1 & 2 & 3 & -4 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_1+R_3} \begin{pmatrix} 2 & 1 & 2 & 2 & -4 & k \\ 0 & 1 & 0 & 1 & 0 & -k-3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix} \xrightarrow{-R_3+R_2} \begin{pmatrix} 2 & 1 & 2 & 2 & -4 & k \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix}$$

Any further row reduction will not affect the 2nd row.

The equation corresponding to the 2nd row is

$$0x_1 + 1x_2 + 0x_3 + 0x_4 + 0x_5 = -3\,.$$

So $x_2 = \underline{\underline{-3}}$.

30) In the following, k is some number:

$$\begin{aligned} 2x_1 + 1x_2 + 2x_3 + 2x_4 - 4x_5 &= k \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 4x_5 &= -3 \\ 2x_1 + 1x_2 + 2x_3 + 3x_4 - 4x_5 &= 0\,. \end{aligned}$$

Solve this system for x_1 in terms of the free variables.

This is the same system as in the previous problem. Continue with the row reduction.

$$\begin{pmatrix} 2 & 1 & 2 & 2 & -4 & k \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix} \xrightarrow{-2R_2+R_1} \begin{pmatrix} 2 & 0 & 2 & 2 & -4 & k+6 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix}$$

$$\xrightarrow{-2R_3+R_1} \begin{pmatrix} 2 & 0 & 2 & 0 & -4 & 3k+6 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix}$$

$$\xrightarrow{1/2\, R_1} \begin{pmatrix} 1 & 0 & 1 & 0 & -2 & 3/2\,k+3 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -k \end{pmatrix}$$

This is now in reduced form. The 1st line corresponds to the equation

$$x_1 + x_3 - 2x_5 = 3/2\,k + 3 \quad \text{or}$$

$$\underline{\underline{x_1 = 3/2\,k + 3 - x_3 + 2x_5}}$$

31) Let a and b be numbers and let

$$A = \begin{pmatrix} a & 10 \\ 20 & b \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Find the (3,2) entry of BA - B.

(A) a (B) $a-1$ (C) 40 (D) $2b$ (E) $2b-2$

(F) none of the others

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} a & 10 \\ 20 & b \end{pmatrix}}_{A} - \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}}_{B} = \begin{pmatrix} a & 10 \\ a & 16 \\ 40 & 2b \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} a-1 & 16 \\ a-1 & 10 \\ 40 & 2b-2 \end{pmatrix}$$

(3,2) entry

Answer: $2b-2$

32) Find the entry in row 1 and column three of the matrix A^{-1}, where

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

(A) $3a$ (B) $-3a$ (C) $3+a$ (D) 3 (E) -3

(F) none of the others

Augment this matrix with I_3 and row reduce.

$$\left(\underbrace{\begin{matrix} 1 & a & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \xrightarrow{-aR_2+R_1} \begin{pmatrix} 1 & 0 & 3a & 1 & -a & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{3R_3+R_2} \begin{pmatrix} 1 & 0 & 3a & 1 & -a & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

(3,1) entry

$$\xrightarrow{-3aR_3+R_1} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} 1 & -a & -3a \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{matrix}}_{A^{-1}}\right)$$

Answer: $\underline{\underline{-3a}}$

33) If the following augmented matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b & 10 \\ 2 & 4 & 6 & 10 & 4a & 4b & 40 \\ 1 & 2 & 3 & 5 & a & b & 10 \end{pmatrix}.$$

is row reduced, and if $a \neq 0$ the (1,5) entry will be:

(A) $4a$ (B) 1 (C) 0 (D) $3a$ (E) $-3a$

(F) none of the others

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a & b & 10 \\ 2 & 4 & 6 & 10 & 4a & 4b & 40 \\ 1 & 2 & 3 & 5 & a & b & 10 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 & 5 & a & b & 10 \\ 2 & 4 & 6 & 10 & 4a & 4b & 40 \\ 0 & 0 & 0 & 0 & a & b & 10 \end{pmatrix}$$

$$\xrightarrow{-2R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 & 5 & a & b & 10 \\ 0 & 0 & 0 & 0 & 2a & 2b & 20 \\ 0 & 0 & 0 & 0 & a & b & 10 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 + R_3} \begin{pmatrix} 1 & 2 & 3 & 5 & a & b & 10 \\ 0 & 0 & 0 & 0 & 2a & 2b & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & 3 & 5 & a & b & 10 \\ 0 & 0 & 0 & 0 & a & b & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-R_2 + R_1} \begin{pmatrix} 1 & 2 & 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{a}R_2} \begin{pmatrix} 1 & 2 & 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & b/a & 10/a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(1,5) entry

Answer: 0.